From regular to chaotic motions in Dynamical Systems with applications to asteroid and debris dynamics

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STARDUST

PUSHING THE BOUNDARIES OF SPACE RESEARCH TO SAVE OUR FUTURE



Outline

1. Introduction

- 2. Regular, periodic and chaotic dynamics
- 2.1 Hamiltonian formalism
- 2.2 Conservative Standard Map
- 2.3 Dissipative Standard Map
- 3. Dynamical behaviors
- 4. Models for NEO and space debris
- 4.1 Rotational dynamics
- 4.2 Spin-orbit problem
- 4.3 Restricted three–body problem
- 4.4 Space debris
- 5. Dynamical numerical methods
- 5.1 Poincaré maps
- 5.2 Lyapunov exponents
- 5.3 FLI
- 6. Basics of perturbation theory
- 7. Conclusions and perspectives

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• Aim of the lecture: to introduce regular and chaotic dynamics, to present orbital and rotational models for NEO and space debris, to provide the main tools to investigate the dynamics.

• MODELS: conservative and dissipative effects



▷ PLANETS:

- *N*-body (planetary) problem
- Poynting-Robertson effect, Stokes drag (primordial solar nebula), tides

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- restricted 3-body problem
- tidal torque, Yarkowski/YORP effects
- ▷ SPACECRAFT AND SPACE DEBRIS:
- geopotential, restricted 3-body problem (Sun and Moon), solar radiation pressure
- atmospheric drag, sloshing, mass consumption.

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Nearly-integrable systems of the form

$$\mathcal{H}(y,x) = h(y) + \varepsilon f(y,x) ,$$

where $y \in \mathbb{R}^n$ (actions), $x \in \mathbb{T}^n$ (angles), $\varepsilon > 0$ is a small parameter.

• In the *integrable* approximation $\varepsilon = 0$ Hamilton's equations are solved as

$$\dot{y} = -\frac{\partial h(y)}{\partial x} = 0 \quad \Rightarrow \quad y(t) = y(0) = const.$$

$$\dot{x} = \frac{\partial h(y)}{\partial y} \equiv \omega(y) \quad \Rightarrow \quad x(t) = \omega(y(0)) t + x(0) ,$$

where (y(0), x(0)) are the initial conditions.

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where (y(0), x(0)) are the initial conditions. • If $\varepsilon \neq 0$, then the system is non-integrable:

$$\dot{y} = -\varepsilon \frac{\partial f(y, x)}{\partial x}$$
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where (y(0), x(0)) are the initial conditions.

• In many cases it is useful to consider also *nearly-integrable dissipative* systems, like ($\lambda > 0$ dissipative constant, μ drift term):

$$\dot{y} = -\varepsilon \frac{\partial f(y, x)}{\partial x} - \lambda (y - \mu),$$

$$\dot{x} = \omega(y) + \varepsilon \frac{\partial f(y, x)}{\partial y}.$$

It is described by the equations (discrete analogue of the spin-orbit problem)

$$\begin{aligned} y' &= y + \varepsilon \, g(x) & y \in \mathbb{R} \,, \, x \in \mathbb{T} \\ x' &= x + y' \,, \end{aligned}$$

with $\varepsilon > 0$ perturbing parameter, g = g(x) analytic function.

• Classical (Chirikov) standard map: $g(x) = \sin x$.

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- Classical (Chirikov) standard map: $g(x) = \sin x$.
- Equivalent notation:

$$y_{j+1} = y_j + \varepsilon \sin(x_j)$$

$$x_{j+1} = x_j + y_{j+1} = x_j + y_j + \varepsilon \sin(x_j) \quad \text{for } j \ge 0.$$

• PROPERTIES:

A) SM is integrable for $\varepsilon = 0$, non-integrable for $\varepsilon \neq 0$:

$$y_{j+1} = y_j = y_0$$

$$x_{j+1} = x_j + y_{j+1} = x_j + y_j = x_0 + jy_0 \quad \text{for } j \ge 0, \quad (1)$$

namely y_i is constant and x_i increases by y_0 .

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namely y_j is constant and x_j increases by y_0 .

A1) Case $y_0 = 2\pi \frac{p}{q}$ with p, q integers $(q \neq 0)$. Then, on the line $y = y_0$:

$$x_1 = x_0 + 2\pi \frac{p}{q}, \quad x_2 = x_0 + 4\pi \frac{p}{q}, \dots, x_q = x_0 + 2\pi p = x_0$$
!!!

Therefore, the orbit is PERIODIC with period $2\pi q$ and the interval $[0, 2\pi)$ is spanned *p* times.

A2) Case $y_0 = 2\pi$ -irrational. Then, on the line $y = y_0$, the iterates of x_0 fill densely the line $y = y_0 \rightarrow$ QUASI-PERIODIC MOTIONS (KAM theory): the iterates never come back to the initial condition, but close as you wish after a sufficient number of iterations.

B) The mapping (1) is conservative, since the determinant of the corresponding Jacobian is equal to one; in fact, setting $f_x(x_j) \equiv \frac{\partial f(x_j)}{\partial x}$, the determinant of the Jacobian (1) is equal to

$$\det \begin{pmatrix} 1 & \varepsilon f_x(x_j) \\ 1 & 1 + \varepsilon f_x(x_j) \end{pmatrix} = 1.$$
⁽²⁾

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C) Fixed points are obtained by solving the equations

$$y_{j+1} = y_j$$

 $x_{j+1} = x_j;$

◊ from the first equation $y_{j+1} = y_j + εf(x_j) \Rightarrow f(x_j) = 0$; ◊ from the second equation $x_{j+1} = x_j + y_{j+1} \Rightarrow y_{j+1} = 0 = y_0$; ◊ if f(x) = sin x, fixed points are $(y_0, x_0) = (0, 0)$ and $(y_0, x_0) = (0, \pi)$. D) Linear stability is investigated by computing the first variation:

$$\begin{pmatrix} \delta y_{j+1} \\ \delta x_{j+1} \end{pmatrix} = \begin{pmatrix} 1 & \varepsilon f_x(x_0) \\ 1 & 1 + \varepsilon f_x(x_0) \end{pmatrix} \begin{pmatrix} \delta y_j \\ \delta x_j \end{pmatrix} .$$

The eigenvalues of the linearized system are determined by solving the characteristic equation ($f = \sin x$):

$$\lambda^2 - (2\pm\varepsilon)\lambda + 1 = 0 ,$$

with + for (0, 0) and - for $(0, \pi)$.

 \diamond One eigenvalue associated to (0,0) is greater than one \Rightarrow the fixed point is unstable.

 \diamond For $\varepsilon < 4$ the eigenvalues associated to $(0, \pi)$ are complex conjugate with real part less than one $\Rightarrow (0, \pi)$ is stable.





 $\varepsilon = 0$: the system is integrable, only quasi-periodic curves (lines), a stable equilibrium point at $(0, \pi)$ and an unstable at (0, 0).



 $\varepsilon = 0.1$: switch on the perturbation, the system is non-integrable, the quasi-periodic (KAM) curves are distorted, the stable point $(0, \pi)$ is surrounded by elliptic islands.





 $\varepsilon = 0.6$: increasing the perturbation, the amplitude of the islands increases, the chaotic region around the unstable point is larger (what is chaos???).



 $\varepsilon = 0.9$: for a large perturbation, a lot of chaos, a few quasi-periodic curves, islands around higher-order periodic orbits.



 $\varepsilon = 1$: very large perturbation, no more quasi-periodic curves.

Dissipative Standard Map:

It is described by the equations (discrete analogue of the spin-orbit problem with tidal torque)

$$\begin{array}{ll} y' &=& \lambda y + \mu + \varepsilon \; g(x) & \qquad y \in \mathbb{R} \; , \; x \in \mathbb{T} \\ x' &=& x + y' \; , & \qquad \lambda, \mu, \varepsilon \in \mathbb{R} \; , \quad \varepsilon \geq 0 \; , \end{array}$$

 $0 < \lambda < 1$ dissipative parameter, $\mu = drift$ parameter.

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 $0 < \lambda < 1$ dissipative parameter, $\mu = drift$ parameter.

• PROPERTIES:

A) $\lambda = 1, \mu = 0$ conservative SM.

B) $\lambda \neq 1$, dissipative, since the determinant of the Jacobian amounts to λ . *C*) The drift μ plays a very important role. In fact, consider $\varepsilon = 0$ and look for an invariant solution, such that

$$y' = y \quad \Rightarrow \quad \lambda y + \mu = y \quad \Rightarrow \quad y = \frac{\mu}{1 - \lambda}$$

If $\mu = 0$, then y = 0!







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From the previous examples, we have:

• Periodic motion: a solution of the equations of motion which retraces its own steps after a given interval of time, called *period*.

• Quasi-periodic motion: a solution of the equations of motion which comes indefinitely close to its initial conditions at regular intervals of time, though ever exactly retracing itself.

• **Regular motion**: we will refer to periodic or quasi-periodic orbits as *regular* motions.

• Chaotic motion: irregular motion showing an *extreme sensitivity* to the choice of the initial conditions.

 \diamond The divergence of the orbits will be measured by the *Lyapunov exponents* or by the *FLI*.

 \Diamond Chaotic motions are unpredictable, but not necessarily unstable.
- **Resonance:** commensurability relation among the revolution and/or rotation periods.
- Rotational dynamics: $(spin-orbit problem) \rightarrow e.g.$ the Moon rotating around its spin-axis and orbiting around the Earth \rightarrow commensurability between the period of rotation and revolution.
- Orbital dynamics: $(three-body problem) \rightarrow e.g.$ asteroid-Sun-Jupiter \rightarrow commensurability between the orbital period of the asteroid and of Jupiter.

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A satellite/asteroid orbiting around a planet and rotating about its spin axis.

• Important dissipative effect: **tidal torque**, due to the non-rigidity, and **Yarkovsky/YORP effect:** due to the joint action of solar lighting and rotation of the body (the rotation causes that the re-emission of the absorbed radiation occurs along a direction different from that of the Sun, thus provoking a variation of the angular momentum and therefore of the orbit).

Spin-orbit problem

• Spin–orbit problem: triaxial satellite/asteroid S (with A < B < C) moving on a Keplerian orbit around a central planet P, assuming that the spin–axis is perpendicular to the orbit plane and coincides with the shortest physical axis.

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• Equation of motion:

$$\ddot{x} + \varepsilon (\frac{a}{r})^3 \sin(2x - 2f) = 0$$
, $\varepsilon = \frac{3}{2} \frac{B - A}{C}$,

corresponding to a 1-dim, time-dependent Hamiltonian:

$$\mathcal{H}(y,x,t) = \frac{y^2}{2} - \frac{\varepsilon}{2} \left(\frac{a}{r(t)}\right)^3 \cos(2x - 2f(t)) .$$

• Spin–orbit resonance of order p/q (p, q integers): $\frac{T_{rev}}{T_{rot}} = \frac{p}{q}$.

$$T_{rev} = \frac{2\pi}{\omega_{rev}}, \quad T_{rot} = \frac{2\pi}{\omega_{rot}} \quad \Rightarrow \quad \frac{\omega_{rot}}{\omega_{rev}} = \frac{p}{q} \quad \Rightarrow \quad \frac{\dot{x}}{\dot{\ell}} = \frac{p}{q},$$

where ℓ is the mean anomaly of the Keplerian orbit.

The **Moon** and all evolved satellites always point the same face to the host planet: 1:1 resonance, i.e. 1 rotation = 1 revolution. Asteroids/NEOs might have different rotational periods. Only exception: **Mercury** in a 3:2 spin–orbit resonance (3 rotations = 2 revolutions).

• Spin–orbit equation with tidal torque:

$$\ddot{x} + \varepsilon \left(\frac{a}{r}\right)^3 \sin(2x - 2f) = -\lambda(\dot{x} - \mu) ,$$

where $\mu = \mu(e)$ and $\lambda = K \tilde{\lambda}(e)$ with *K* depending on the physical properties of the satellite/asteroid.

Relation between the standard map and the spin-orbit model

Conservative spin-orbit model:

$$\ddot{x} = \varepsilon g(x,t)$$
, $g(x,t) = -\left(\frac{a}{r(t)}\right)^3 \sin(2x - 2f(t))$

which can be written as

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \varepsilon g(x,t) , \end{aligned}$$

or

$$\dot{y} = \varepsilon g(x,t)$$

 $\dot{x} = y$.

Integrate with a symplectic (first order) Euler's method with step-size h:

 $y_{j+1} = y_j + \varepsilon g(x_j, t)h$ $x_{j+1} = x_j + y_{j+1}h$

with $t_{j+1} = t_j + h \Rightarrow$

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... and dissipative SM with spin-orbit model with tides

Dissipative spin-orbit model:

$$\ddot{x} = \varepsilon g(x,t) - \lambda(\dot{x} - \mu)$$
, $g(x,t) = -\left(\frac{a}{r(t)}\right)^3 \sin(2x - 2f(t))$,

which can be written as

$$\dot{x} = y \dot{y} = \varepsilon g(x,t) - \lambda(y-\mu) ,$$

or

$$\dot{y} = \varepsilon g(x, t) - \lambda (y - \mu) \dot{x} = y.$$

Integrate with a symplectic (first order) Euler's method with step-size h:

$$y_{j+1} = y_j + \varepsilon g(x_j, t)h - \lambda(y_j - \mu) = (1 - \lambda)y_j + \lambda \mu + \varepsilon g(x_j, t)h$$
$$x_{j+1} = x_j + y_{j+1}h$$
with $t_{j+1} = t_j + h \Rightarrow$ dissipative standard map!

From regular to chaotic motions

• Consider the motion of a small body (with negligible mass) under the gravitational influence of two primaries, moving on Keplerian orbits about their common barycenter (*restricted* problem).

• Assume that the orbits of the primaries are circular and that all bodies move on the same plane: *planar, circular, restricted three–body problem* (PCR3BP).

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Adopting suitable normalized units and action-angle Delaunay variables (L, G) ∈ ℝ², (ℓ, g) ∈ T², we obtain a 2 d.o.f. Hamiltonian function:

$$\mathcal{H}(L,G,\ell,g) = -\frac{1}{2L^2} - G + \varepsilon R(L,G,\ell,g) \; .$$

- ε primaries' mass ratio ($\varepsilon = 0$ Keplerian motion).
- Actions: $L = \sqrt{a}$, $G = L\sqrt{1-e^2}$.
- Angles: the mean anomaly ℓ , $g = \tilde{\omega} t$ with $\tilde{\omega}$ argument of perihelion.

• $R = R(L, G, \ell, g)$ represents the interaction with P_3 (use a trigonometric approximation).

• **Resonance:** let T_{ast} , T_{Jup} be the orbital period; a p : q mean motion resonance (p, q integers) occurs when $T_{Jup}/T_{ast} = p/q$.

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• Dissipative effects:

 \diamond **Stokes drag:** collision of particles with the molecules of the gas nebula during the formation of the planetary system

 \diamond **Poynting–Robertson effect:** due to the absorption and re-emission of the solar radiation, the velocity of a dust particle decreases

• In an inertial reference frame (O, ξ, η) : $P = (\xi, \eta), P_1 = (\xi_1, \eta_1) \text{ mass } 1 - m, P_2 = (\xi_2, \eta_2) \text{ mass } m.$

• Equations of motion:

$$\begin{split} \ddot{\xi} &= m \frac{\xi_1 - \xi}{r_1^3} + (1 - m) \frac{\xi_2 - \xi}{r_2^3} \\ \ddot{\eta} &= m \frac{\eta_1 - \eta}{r_1^3} + (1 - m) \frac{\eta_2 - \eta}{r_2^3} \,, \end{split}$$

where $r_1 = \sqrt{(\xi_1 - \xi)^2 + (\eta_1 - \eta)^2}$, $r_2 = \sqrt{(\xi_2 - \xi)^2 + (\eta_2 - \eta)^2}$ are the distances from the primaries.

• Synodic frame rotating with the angular velocity of the primaries:

 $\xi = x \cos t - y \sin t$ $\eta = x \sin t + y \cos t.$

• New distances from the primaries:

$$r_1 = \sqrt{(x+m)^2 + y^2}$$
, $r_2 = \sqrt{(x-1+m)^2 + y^2}$.

- Location of the primaries: $P_1 = (-m, 0), P_2 = (1 m, 0).$
- Equations of motion in the synodic frame:

$$\ddot{x} = 2\dot{y} + x - (1 - m)\frac{x + m}{r_1^3} - m\frac{x - 1 + m}{r_2^3}$$
$$\ddot{y} = -2\dot{x} + y - (1 - m)\frac{y}{r_1^3} - m\frac{y}{r_2^3}.$$

Dissipative three–body problem

• Equations of motion for P = (x, y) in a synodic frame with primaries $P_1 = (-m, 0), P_2 = (1 - m, 0)$:

$$\begin{aligned} \ddot{x} &= 2\dot{y} + x - (1-m)\frac{x+m}{r_1^3} - m\frac{x-1+m}{r_2^3} + F_x \\ \ddot{y} &= -2\dot{x} + y - (1-m)\frac{y}{r_1^3} - m\frac{y}{r_2^3} + F_y , \end{aligned}$$

where $r_1^2 = (x + m)^2 + y^2$, $r_2^2 = (x - 1 + m)^2 + y^2$, K = dissipative constant,

$$\begin{array}{lll} (F_x,F_y) &=& -K(\dot{x}-y,\dot{y}+x) & (linear) \\ (F_x,F_y) &=& -K(\dot{x}-y+\alpha\Omega y,\dot{y}+x-\alpha\Omega x) & (Stokes) \\ (F_x,F_y) &=& -\frac{K}{r_1^2}(\dot{x}-y,\dot{y}+x) & (PR) \,. \end{array}$$

 $\Omega = \Omega(r) \equiv r^{-3/2}$ is the Keplerian angular velocity at distance $r = \sqrt{x^2 + y^2}$, $\alpha \in [0, 1)$ ratio between the gas and Keplerian velocities.

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From regular to chaotic motions

• LEO (0-2 000 km): affected by Earth, air drag, Earth's oblateness, Moon, Sun, SRP.

• MEO (2000-30000 km): affected by Earth, Earth's oblateness, Moon, Sun, SRP.

 \diamondsuit GPS at 26 560 km with a period of 12^h (2:1 resonance).

• GEO (>30 000 km): affected by Earth, Earth's oblateness, Moon, Sun, SRP. \Diamond Geostationary ring at 42 164 km with a period of 24^{*h*} (1:1 resonance).

• **Debris** *p* : *q* **resonance:** *p* times the orbital period of the debris is equal to *q* times the rotational period of the Earth.

In a quasi-inertial frame centered in the Earth, the equations of motion are provided by the sum of the contributions of the Earth's gravitational influence, including the oblateness effect, the solar attraction, the lunar attraction as well as the sum of the contributions of the non-gravitational forces (SRP, air drag) denoted as \mathbf{a}_{ng} :

$$\ddot{\mathbf{r}} = -G \int_{V_E} \rho(\mathbf{r}_p) \frac{\mathbf{r} - \mathbf{r}_p}{|\mathbf{r} - \mathbf{r}_p|^3} dV_E - Gm_S \left(\frac{\mathbf{r} - \mathbf{r}_S}{|\mathbf{r} - \mathbf{r}_S|^3} + \frac{\mathbf{r}_S}{|\mathbf{r}_S|^3}\right) - Gm_M \left(\frac{\mathbf{r} - \mathbf{r}_M}{|\mathbf{r} - \mathbf{r}_M|^3} + \frac{\mathbf{r}_M}{|\mathbf{r}_M|^3}\right) + \mathbf{a}_{ng} ,$$

where $\rho(\mathbf{r}_p)$ is the density at some point \mathbf{r}_p inside the volume V_E of the Earth, m_S , m_M are the masses of the Sun and the Moon, respectively, \mathbf{r}_S , \mathbf{r}_M are the distance vectors of the Sun and the Moon with respect to the Earth's center.

In cartesian coordinates the components are:

$$\begin{aligned} \ddot{x} &= -\frac{GM_{Ex}}{r^{3}} + V_{geo,x}(x, y, z) \\ &- Gm_{S} \Big(\frac{x - x_{S}}{|\mathbf{r} - \mathbf{r}_{S}|^{3}} + \frac{x_{S}}{r_{S}^{3}} \Big) - Gm_{M} \Big(\frac{x - x_{M}}{|\mathbf{r} - \mathbf{r}_{M}|^{3}} + \frac{x_{M}}{r_{M}^{3}} \Big) + a_{1ng} \\ \ddot{y} &= -\frac{GM_{Ey}}{r^{3}} + V_{geo,y}(x, y, z) \\ &- Gm_{S} \Big(\frac{y - y_{S}}{|\mathbf{r} - \mathbf{r}_{S}|^{3}} + \frac{y_{S}}{r_{S}^{3}} \Big) - Gm_{M} \Big(\frac{y - y_{M}}{|\mathbf{r} - \mathbf{r}_{M}|^{3}} + \frac{y_{M}}{r_{M}^{3}} \Big) + a_{2ng} \\ \ddot{z} &= -\frac{GM_{Ez}}{r^{3}} + V_{geo,z}(x, y, z) \\ &- Gm_{S} \Big(\frac{z - z_{S}}{|\mathbf{r} - \mathbf{r}_{S}|^{3}} + \frac{z_{S}}{r_{S}^{3}} \Big) - Gm_{M} \Big(\frac{z - z_{M}}{|\mathbf{r} - \mathbf{r}_{M}|^{3}} + \frac{z_{M}}{r_{M}^{3}} \Big) + a_{3ng} . \end{aligned}$$

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• V_{geo} in terms of spherical coordinates and harmonics:

$$V_{geo}(r,\phi,\lambda) = \frac{GM_E}{r} \sum_{i=0}^{\infty} (\frac{R_E}{r})^i \sum_{k=0}^i P_i^k(\sin\phi) \left(C_{ik}\cos k\lambda + S_{ik}\sin k\lambda\right),$$

where the quantities P_i^k are defined in terms of the Legendre polynomials:

$$P_n(x) \equiv \frac{1}{2^n n!} \frac{d^n}{dx^n} \{ (x^2 - 1)^n \}$$

$$P_n^m(x) \equiv (1 - x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} \{ P_n(x) \} ,$$

while C_{ik} , S_{ik} are defined as

$$C_{nm} \equiv \frac{2-\delta_{0m}}{M_E} \frac{(n-m)!}{(n+m)!} \int_{V_E} (\frac{r_p}{R_E})^n P_n^m(\sin\phi_p) \cos(m\lambda_p)\rho(\mathbf{r}_p) dV_E$$

$$S_{nm} \equiv \frac{2-\delta_{0m}}{M_E} \frac{(n-m)!}{(n+m)!} \int_{V_E} (\frac{r_p}{R_E})^n P_n^m(\sin\phi_p) \sin(m\lambda_p)\rho(\mathbf{r}_p) dV_E ,$$

 $(r_p, \lambda_p, \phi_p) =$ spherical coordinates of a point *P* inside the Earth. • $J_2 = -C_{20}, J_{22} = \sqrt{C_{22}^2 + S_{22}^2}$.

• SRP contribution:

$$\mathbf{F}_{srp} = C_r P_r a_S^2 \left(\frac{A}{m}\right) \frac{\mathbf{r} - \mathbf{r}_S}{|\mathbf{r} - \mathbf{r}_S|^3} ,$$

where C_r is the reflectivity coefficient, depending on the optical properties of the space debris surface, P_r is the radiation pressure for an object located at $a_S = 1 AU$, $\frac{A}{m}$ is the area-to-mass ratio with A being the cross-section of the space debris, \mathbf{r}_S is (as above) the geocentric position of the Sun.

• Atmospheric drag: the acceleration of the satellite due to atmospheric drag can be modeled as

$$\mathbf{a}_d = -\frac{C_D}{2}\rho \,\frac{A}{m} \,V^2 \,\mathbf{e_v} \;,$$

where *V* is the velocity of the debris relative to the atmosphere, $\mathbf{e}_{\mathbf{v}}$ is the unit vector of the debris velocity relative to the atmosphere, C_D is the drag coefficient which can be assumed within $2 \le C_D \le 2.5$, where $C_D = 2$ holds for spherical satellites.

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• The Poincaré map reduces the study of a continuous system to that of a discrete mapping.

• Consider the *n*-dimensional differential system

 $\underline{\dot{z}} = \underline{f}(\underline{z}) , \qquad \underline{z} \in \mathbf{R}^n ,$

where $f = f(\underline{z})$ is a generic regular vector field.

• Let $\underline{\Phi}(t;\underline{z}_0)$ be the flow at time *t* with initial condition \underline{z}_0 .

• Let Σ be an (n-1)-dimensional hypersurface, the *Poincaré section*, transverse to the flow, which means that if $\underline{\nu}(\underline{z})$ denotes the unit normal to Σ at \underline{z} , then $f(\underline{z}) \cdot \underline{\nu}(\underline{z}) \neq 0$ for any \underline{z} in Σ .

• Poincaré map of the spin–orbit model:

$$\dot{x} = y$$

$$\dot{y} = -\varepsilon (\frac{a}{r})^3 \sin(2x - 2f)$$

with

$$r = a(1 - e\cos u)$$

$$\tan \frac{f}{2} = \sqrt{\frac{1 + e}{1 - e}} \tan \frac{u}{2}$$

$$\ell = u - e\sin u$$

$$\ell = nt + \ell_0.$$

• One-dimensional, time-dependent (2π -periodic in time):

$$\dot{x} = y$$

 $\dot{y} = \varepsilon g(x, t)$

• Poincaré maps of the spin-orbit problem taking the intersections with $t = 2\pi k, k \in \mathbb{Z}_+$ for $\varepsilon = 0.024, 0.1, 0.4$.



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- Poincaré map of the 3-body problem, needs:

 \[
 regularization theory in the neighborhood of close encounters!
 \]
- Hénon–Heiles model: 2–dimensional nonlinear, non–integrable system describing the motion of stars around the galactic center:

$$H(\dot{x}, \dot{y}, x, y) = H = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + Ax^2 + By^2) + x^2y - \frac{1}{3}\epsilon y^3 .$$

• Poincaré maps of the Hénon–Heiles model E = 0.8333 and E = 0.125:



AUTHOR: Fabien Gachet.

• Lyapunov exponents provide the divergence of nearby orbits.

• Quantitatively, two nearby trajectories at initial distance $\delta_{\underline{z}_0}$ diverge at a rate given by (within the linearized approximation)

 $|\delta \underline{z}(t)| \approx e^{\lambda t} |\delta \underline{z}(0)|$,

where λ is the Lyapunov exponent.

• The rate of separation can be different in different directions \rightarrow there is a spectrum of Lyapunov exponents equal in number to the dimension of the phase space.

• The largest Lyapunov exponent is called *Maximal Lyapunov exponent* and a positive value gives an indication of chaos. It can be computed as

$$\lambda = \lim_{t \to \infty} \lim_{\delta \underline{z}(0) \to 0} \frac{1}{t} \ln \frac{|\delta \underline{z}(t)|}{|\delta \underline{z}(0)|} .$$

- Fast Lyapunov Indicator (FLI) is obtained as the value of the MLE at a fixed time, say *T*.
- A comparison of the FLIs as the initial conditions are varied allows one to distinguish between different kinds of motion (regular, resonant or chaotic).
- Consider $\underline{\dot{z}} = \underline{f}(\underline{z}), \underline{z} \in \mathbf{R}^n$ and let the variational equations be

$$\underline{\dot{\nu}} = \left(\frac{\partial \underline{f}(\underline{z})}{\partial \underline{z}}\right) \underline{\nu} \,.$$

• Definition of the FLI: given the initial conditions $\underline{z}(0) \in \mathbf{R}^n$, $\underline{v}(0) \in \mathbf{R}^n$, the FLI at time $T \ge 0$ is provided by the expression

$$FLI(\underline{z}(0), \underline{v}(0), T) \equiv \sup_{0 < t \le T} \log ||\underline{v}(t)||.$$

• FLI for the spin-orbit problem in the x, $p_x = y$ plane: Blue = regular motions, red/green = chaotic dynamics



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... and in the parameter space ε versus p_x (with $x_0 = 0$) for Mercury (left) and Moon (right)



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Basics of perturbation theory

Perturbation theory is based on the implementation of a canonical transformation to find the solution within a better degree of approximation.
For an *n*-dimensional nearly-integrable Hamiltonian function

$$\mathcal{H}(\underline{I},\underline{\varphi}) = h(\underline{I}) + \varepsilon f(\underline{I},\underline{\varphi}) , \qquad (3)$$

let the frequency vector be defined as

$$\underline{\omega}(\underline{I}) \equiv \frac{\partial h(\underline{I})}{\partial \underline{I}} \; .$$

• Construct a canonical transformation $C : (\underline{I}, \underline{\varphi}) \to (\underline{I}', \underline{\varphi}')$, such that \mathcal{H} takes the form

$$\mathcal{H}'(\underline{I}', \underline{\varphi}') \;=\; \mathcal{H} \circ \mathcal{C}(\underline{I}, \underline{\varphi}) \;\equiv\; h'(\underline{I}') + \varepsilon^2 f'(\underline{I}', \underline{\varphi}') \;,$$

where h' and f' denote, respectively, the new unperturbed Hamiltonian and the new perturbing function.

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- The result is obtained through the following steps:
- *i*) define a suitable canonical transformation close to the identity,
- ii) perform a Taylor series expansion in the perturbing parameter,
- *iii*) require that the change of variables removes the dependence on the angles up to second order terms,
- iv) expand in Fourier series to construct the explicit form of the canonical transformation.

i) Define a close–to–identity change of variables with generating function $\underline{I}' \cdot \varphi + \varepsilon \Phi(\underline{I}', \varphi)$ providing

$$\underline{I} = \underline{I}' + \varepsilon \frac{\partial \Phi(\underline{I}', \underline{\varphi})}{\partial \underline{\varphi}}$$

$$\underline{\varphi}' = \underline{\varphi} + \varepsilon \frac{\partial \Phi(\underline{I}', \underline{\varphi})}{\partial \underline{I}'} ,$$

where $\Phi = \Phi(\underline{I}', \underline{\varphi})$ is unknown.

ii) Split f as $f(\underline{I}, \underline{\varphi}) = \overline{f}(\underline{I}) + \tilde{f}(\underline{I}, \underline{\varphi})$, where $\overline{f}(\underline{I})$ is the average and $\tilde{f}(\underline{I}, \underline{\varphi})$ is the remainder.

Inserting the transformation and expanding in Taylor series around $\varepsilon = 0$ up to the second order, one gets

$$\begin{split} h(\underline{I}' + \varepsilon \frac{\partial \Phi(\underline{I}', \underline{\varphi})}{\partial \underline{\varphi}}) + \varepsilon f(\underline{I}' + \varepsilon \frac{\partial \Phi(\underline{I}', \underline{\varphi})}{\partial \underline{\varphi}}, \varphi) \\ = & h(\underline{I}') + \underline{\omega}(\underline{I}') \cdot \varepsilon \frac{\partial \Phi(\underline{I}', \underline{\varphi})}{\partial \underline{\varphi}} + \varepsilon \overline{f}(\underline{I}') + \varepsilon \widetilde{f}(\underline{I}', \underline{\varphi}) + O(\varepsilon^2) \;. \end{split}$$

iii) Write the new Hamiltonian as

$$\mathcal{H}'(\underline{I}',\underline{\varphi}) = h(\underline{I}') + \varepsilon \overline{f}(\underline{I}') + \varepsilon \left(\underline{\omega}(\underline{I}') \cdot \frac{\partial \Phi(\underline{I}',\underline{\varphi})}{\partial \underline{\varphi}} + \varepsilon \widetilde{f}(\underline{I}',\underline{\varphi})\right) + O(\varepsilon^2)$$

• The transformed Hamiltonian is integrable up to second order in ε provided Φ satisfies:

$$\underline{\omega}(\underline{I}') \cdot \frac{\partial \Phi(\underline{I}', \underline{\varphi})}{\partial \underline{\varphi}} + \tilde{f}(\underline{I}', \underline{\varphi}) = \underline{0} .$$

• The new unperturbed Hamiltonian becomes

$$h'(\underline{I}') = h(\underline{I}') + \varepsilon \overline{f}(\underline{I}')$$
.

iv) Explicit expression of Φ obtained expanding in Fourier series as

$$\begin{split} \Phi(\underline{I}',\underline{\varphi}) &= \sum_{\underline{m}\in\mathbf{Z}^n\setminus\{\underline{0}\}} \hat{\Phi}_{\underline{m}}(\underline{I}') \; e^{i\underline{m}\cdot\underline{\varphi}} \;, \\ \tilde{f}(\underline{I}',\underline{\varphi}) &= \sum_{\underline{m}\in\mathcal{I}} \hat{f}_{\underline{m}}(\underline{I}') \; e^{i\underline{m}\cdot\underline{\varphi}} \;, \end{split}$$

where \mathcal{I} is a suitable set of integer vectors associated to f.

• Inserting the expansion in the equation for Φ :

$$i \sum_{\underline{m} \in \mathbf{Z}^n \setminus \{\underline{0}\}} \underline{\omega}(\underline{I}') \cdot \underline{m} \, \hat{\Phi}_{\underline{m}}(\underline{I}') \, e^{i\underline{m} \cdot \underline{\varphi}} = -\sum_{\underline{m} \in \mathcal{I}} \hat{f}_{\underline{m}}(\underline{I}') \, e^{i\underline{m} \cdot \underline{\varphi}} \,,$$

which provides

$$\hat{\Phi}_{\underline{m}}(\underline{I}') = -\frac{\hat{f}_{\underline{m}}(\underline{I}')}{i \,\underline{\omega}(\underline{I}') \cdot \underline{m}} \,.$$
• The generating function is given by

$$\Phi(\underline{I}',\underline{\varphi}) = i \sum_{\underline{m}\in\mathcal{I}} \frac{\hat{f}_{\underline{m}}(\underline{I}')}{\underline{\omega}(\underline{I}')\cdot\underline{m}} e^{i\underline{m}\cdot\underline{\varphi}} \,.$$

- The algorithm is constructive!
- The function Φ is well defined unless there exists an integer vector $\underline{m} \in \mathcal{I}$ such that

$$\underline{\omega}(\underline{I}') \cdot \underline{m} = 0 \; .$$

This is the classical problem of the small divisors and we must ASSUME that

$$\underline{\omega}(\underline{I}') \cdot \underline{m} \neq 0$$
 for all $\underline{m} \in \mathcal{I}$.

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 - regularization theory near collisions or close–encounters.

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