Classical methods of orbit determination

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Outline

(1) The orbit determination problem

- preliminary orbits:
  Laplace's method
  Gauss' method
- least squares solutions

(2) Multiple solutions

- Charlier's theory
- generalization of the theory

The discovery of Ceres

Giuseppe Piazzi (1746-1826)

On January 1, 1801 G. Piazzi detected Ceres, the first asteroid.
He could follow up the asteroid in the sky for about 1 month, collecting 21 observations forming an arc of ~ 3 degrees.

- Problem: find in which part of the sky we should observe to recover Ceres;
- Orbit determination: given the observations of a celestial body, compute its orbital elements.

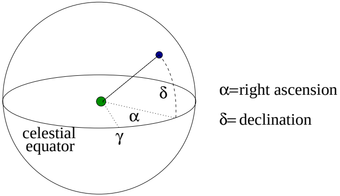
Orbit determination methods

Ceres was recovered in 1802 by H. W. Olbers and F. Von Zach following the computations of C. F. Gauss.
Gauss determined an orbit with Piazzi's observations.
Given at least three observations of a Solar system body, his method consists of two steps:

1. computation of a preliminary orbit;
2. application of the least squares method (differential corrections), using the preliminary orbit as a starting guess.

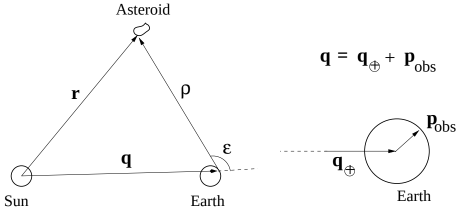
Preliminary orbits:

- Laplace's method (1780)
- Gauss' method (1809)



Optical observation: two angles (α, δ) giving a point on the celestial sphere.
The topocentric distance ρ of the observed body is unknown.

Geocentric and topocentric point of view



$$
\begin{aligned}
\mathbf{r} &= \boldsymbol{\rho} + \mathbf{q} \\
\epsilon &= \textit{co-elongation}
\end{aligned}
$$

Laplace's method

$\boldsymbol{\rho} = \rho\hat{\boldsymbol{\rho}}$ is the geocentric position vector of the observed body,

$$\rho = \|\boldsymbol{\rho}\|, \qquad \hat{\boldsymbol{\rho}} = (\cos\delta\cos\alpha, \cos\delta\sin\alpha, \sin\delta),$$

with α, δ the *right ascension* and *declination*.

$\mathbf{q} = q\hat{\mathbf{q}}$ is the heliocentric position vector of the center of the Earth, with $q = \|\mathbf{q}\|$.

$\mathbf{r} = \mathbf{q} + \boldsymbol{\rho}$ is the heliocentric position of the body.

We use the *arc length* s to parametrize the motion:

$$\eta = \frac{ds}{dt} = \sqrt{\dot{\alpha}^2\cos^2\delta + \dot{\delta}^2} \qquad \textit{proper motion}$$

Laplace's method

Using the moving orthogonal frame

$$\hat{\boldsymbol{\rho}}, \quad \hat{\mathbf{v}} = \frac{d\hat{\boldsymbol{\rho}}}{ds}, \quad \hat{\mathbf{n}} = \hat{\boldsymbol{\rho}} \times \hat{\mathbf{v}},$$

we introduce the *geodesic curvature* κ by

$$\frac{d\hat{\mathbf{v}}}{ds} = -\hat{\boldsymbol{\rho}} + \kappa\hat{\mathbf{n}}.$$

The acceleration of $\boldsymbol{\rho}$ is given by

$$\frac{d^2}{dt^2}\boldsymbol{\rho} = (\ddot{\rho} - \rho\eta^2)\hat{\boldsymbol{\rho}} + (\rho\dot{\eta} + 2\dot{\rho}\eta)\hat{\mathbf{v}} + (\rho\eta^2\kappa)\hat{\mathbf{n}}.$$

On the other hand we have

$$\frac{d^2}{dt^2}\boldsymbol{\rho} = \frac{d^2}{dt^2}(\mathbf{r} - \mathbf{q})$$

where, according to the equations of the two-body motion,

$$\frac{d^2}{dt^2}\mathbf{r} = -\frac{\mu}{r^3}\mathbf{r}\,, \qquad \frac{d^2}{dt^2}\mathbf{q} = -\frac{\mu + \mu_\oplus}{q^3}\mathbf{q}\,,$$

with $r = \|\mathbf{r}\|$ and μ, $\mu_\oplus$ the masses of the Sun and of the Earth respectively.

Laplace's method

From three or more observations (α_i, δ_i) of a celestial body at times $t_i, i = 1, 2, 3\ldots$ we can interpolate for $\alpha, \delta, \dot{\alpha}, \dot{\delta}$ at a mean time $\bar{t}$.

Neglecting the mass of the Earth and projecting the equation of motion on $\hat{\mathbf{n}}$ at time $\bar{t}$ we obtain

$$\mathcal{C}\frac{\rho}{q} = 1 - \frac{q^3}{r^3} \qquad \text{where} \qquad \mathcal{C} = \frac{\eta^2 \kappa q^3}{\mu(\hat{\mathbf{q}} \cdot \hat{\mathbf{n}})}, \tag{1}$$

where $\rho, q, r, \eta, \hat{\mathbf{q}}, \hat{\mathbf{n}}, \mathcal{C}$ are the values at time $\bar{t}$.
(1) is the dynamical equation of Laplace's method.

Here ρ and r are unknowns, while the other quantities are obtained by interpolation.

Using (1) and the geometric equation

$$r^2 = q^2 + \rho^2 + 2q\rho\cos\epsilon,$$

with $\cos\epsilon = \mathbf{q}\cdot\boldsymbol{\rho}/(q\rho)$ interpolated at time $\bar{t}$, we can write a polynomial equation of degree eight for r by eliminating the geocentric distance ρ:

$$\mathcal{C}^2 r^8 - q^2(\mathcal{C}^2 + 2\mathcal{C}\cos\epsilon + 1)r^6 + 2q^5(\mathcal{C}\cos\epsilon + 1)r^3 - q^8 = 0.$$

Projecting the equation of motion on $\hat{\mathbf{v}}$ yields

$$\rho\dot{\eta} + 2\dot{\rho}\eta = \mu(\mathbf{q}\cdot\hat{\mathbf{v}})\left(\frac{1}{q^3} - \frac{1}{r^3}\right). \qquad (2)$$

We can use equation (2) to compute $\dot{\rho}$ from the values of r, ρ found by the geometric and dynamical equations.

Gauss' method naturally deals with topocentric observations.

This method uses three observations (α_i, δ_i), $i = 1, 2, 3$, related to heliocentric positions of the observed body

$$\mathbf{r}_i = \boldsymbol{\rho}_i + \mathbf{q}_i ,$$

at times t_i, with $t_1 < t_2 < t_3$.

Here $\boldsymbol{\rho}_i$ denotes the topocentric position of the observed body, and $\mathbf{q}_i$ is the heliocentric position of the observer.

We assume that $t_i - t_j$ is much smaller than the period of the orbit.

We assume the coplanarity condition

$$\lambda_1 \mathbf{r}_1 - \mathbf{r}_2 + \lambda_3 \mathbf{r}_3 = 0 \qquad \lambda_1, \lambda_3 \in \mathbb{R}. \tag{3}$$

The vector product of both members of (3) with $\mathbf{r}_i, i = 1, 3$, together with the projection along the direction $\hat{\mathbf{c}}$ of the angular momentum yields

$$\lambda_1 = \frac{\mathbf{r}_2 \times \mathbf{r}_3 \cdot \hat{\mathbf{c}}}{\mathbf{r}_1 \times \mathbf{r}_3 \cdot \hat{\mathbf{c}}}, \qquad \lambda_3 = \frac{\mathbf{r}_1 \times \mathbf{r}_2 \cdot \hat{\mathbf{c}}}{\mathbf{r}_1 \times \mathbf{r}_3 \cdot \hat{\mathbf{c}}}.$$

From the scalar product of both members of (3) with $\hat{\boldsymbol{\rho}}_1 \times \hat{\boldsymbol{\rho}}_3$, using relations $\mathbf{r}_i = \boldsymbol{\rho}_i + \mathbf{q}_i$, we obtain

$$\rho_2(\hat{\boldsymbol{\rho}}_1 \times \hat{\boldsymbol{\rho}}_3 \cdot \hat{\boldsymbol{\rho}}_2) = \hat{\boldsymbol{\rho}}_1 \times \hat{\boldsymbol{\rho}}_3 \cdot (\lambda_1 \mathbf{q}_1 - \mathbf{q}_2 + \lambda_3 \mathbf{q}_3).$$

So far, we have used only the geometry of the orbit.

Now dynamics comes into play.

Development in f, g series: the differences $\mathbf{r}_1 - \mathbf{r}_2$, $\mathbf{r}_3 - \mathbf{r}_2$ are expanded in powers of $t_{i2} = t_i - t_2 = \mathcal{O}(\Delta t)$, $i = 1, 3$.
We can leave only $\mathbf{r}_2$, $\dot{\mathbf{r}}_2$ in the expansion by replacing the second derivative $\ddot{\mathbf{r}}_2$ with $-\mu \mathbf{r}_2 / r_2^3$:

$$\mathbf{r}_i = f_i \mathbf{r}_2 + g_i \dot{\mathbf{r}}_2,$$

where

$$f_i = 1 - \frac{\mu}{2} \frac{t_{i2}^2}{r_2^3} + \mathcal{O}(\Delta t^3), \quad g_i = t_{i2} \left(1 - \frac{\mu}{6} \frac{t_{i2}^2}{r_2^3} \right) + \mathcal{O}(\Delta t^4). \qquad (4)$$

Gauss' method

Using the f, g series we have

$$\begin{aligned} \mathbf{r}_i \times \mathbf{r}_2 &= -g_i \mathbf{c}, \quad i = 1, 3 \\ \mathbf{r}_1 \times \mathbf{r}_3 &= (f_1 g_3 - f_3 g_1) \mathbf{c} \end{aligned}$$

so that

$$\lambda_1 = \frac{g_3}{f_1 g_3 - f_3 g_1}, \qquad \lambda_3 = \frac{-g_1}{f_1 g_3 - f_3 g_1}. \tag{5}$$

Inserting the expressions of f_i, g_i into (5) we obtain

$$\lambda_1 = \frac{t_{32}}{t_{31}} \left[1 + \frac{\mu}{6 r_2^3} (t_{31}^2 - t_{32}^2) \right] + \mathcal{O}(\Delta t^3),$$

$$\lambda_3 = \frac{t_{21}}{t_{31}} \left[1 + \frac{\mu}{6 r_2^3} (t_{31}^2 - t_{21}^2) \right] + \mathcal{O}(\Delta t^3).$$

Now we consider the equation

$$\rho_2(\hat{\boldsymbol{\rho}}_1 \times \hat{\boldsymbol{\rho}}_3 \cdot \hat{\boldsymbol{\rho}}_2) = \hat{\boldsymbol{\rho}}_1 \times \hat{\boldsymbol{\rho}}_3 \cdot (\lambda_1 \mathbf{q}_1 - \mathbf{q}_2 + \lambda_3 \mathbf{q}_3). \qquad (6)$$

Let

$$V = \hat{\boldsymbol{\rho}}_1 \times \hat{\boldsymbol{\rho}}_2 \cdot \hat{\boldsymbol{\rho}}_3.$$

By substituting the expressions for λ_1, λ_3 into (6), using relations

$$t_{31}^2 - t_{32}^2 = t_{21}(t_{31} + t_{32}),$$
$$t_{31}^2 - t_{21}^2 = t_{32}(t_{31} + t_{21}),$$

we can write

$$-V\rho_2 t_{31} = \hat{\boldsymbol{\rho}}_1 \times \hat{\boldsymbol{\rho}}_3 \cdot (t_{32}\mathbf{q}_1 - t_{31}\mathbf{q}_2 + t_{21}\mathbf{q}_3) +$$
$$+\hat{\boldsymbol{\rho}}_1 \times \hat{\boldsymbol{\rho}}_3 \cdot \left[\frac{\mu}{6r_2^3}[t_{32}t_{21}(t_{31}+t_{32})\mathbf{q}_1 + t_{32}t_{21}(t_{31}+t_{21})\mathbf{q}_3]\right] + \mathcal{O}(\Delta t^4).$$

We neglect the $\mathcal{O}(\Delta t^4)$ terms and set

$$A(\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3) = q_2^3\ \hat{\boldsymbol{\rho}}_1 \times \hat{\boldsymbol{\rho}}_3 \cdot [t_{32}\mathbf{q}_1 - t_{31}\mathbf{q}_2 + t_{21}\mathbf{q}_3],$$
$$B(\mathbf{q}_1,\mathbf{q}_3) = \frac{\mu}{6} t_{32} t_{21} \hat{\boldsymbol{\rho}}_1 \times \hat{\boldsymbol{\rho}}_3 \cdot [(t_{31}+t_{32})\mathbf{q}_1 + (t_{31}+t_{21})\mathbf{q}_3].$$

In this way the last equation becomes

$$-\frac{V\ \rho_2\ t_{31}}{B(\mathbf{q}_1,\mathbf{q}_3)}\ q_2^3 = \frac{q_2^3}{r_2^3} + \frac{A(\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3)}{B(\mathbf{q}_1,\mathbf{q}_3)}.$$

Gauss' method

Let

$$\mathcal{C} = \frac{V\, t_{31}\, q_2^4}{B(\mathbf{q}_1, \mathbf{q}_3)}\,, \qquad \gamma = -\frac{A(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3)}{B(\mathbf{q}_1, \mathbf{q}_3)}.$$

We obtain the dynamical equation of Gauss' method:

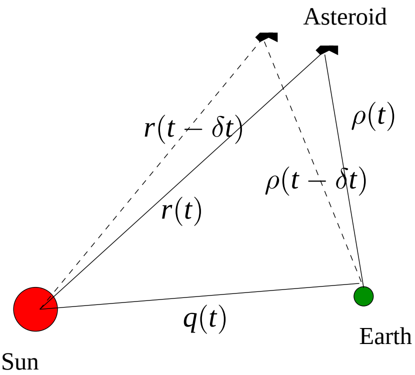
$$\mathcal{C}\,\frac{\rho_2}{q_2} = \gamma - \frac{q_2^3}{r_2^3}. \tag{7}$$

After the possible values for r_2 have been found by (7), coupled with the geometric equation

$$r_2^2 = \rho_2^2 + q_2^2 + 2\rho_2 q_2 \cos\epsilon_2,$$

then the velocity vector $\dot{\mathbf{r}}_2$ can be computed, e.g. from Gibbs' formulas (see Herrick, Chap. 8).

Correction by aberration



with

$$\delta t = \frac{\rho}{c},$$

where ρ is the determined value of the radial distance, and c is the speed of light.

We consider the differential equation

$$\frac{d\mathbf{y}}{dt} = \mathbf{f}(\mathbf{y}, t, \boldsymbol{\mu}) \tag{8}$$

giving the state $\mathbf{y} \in \mathbb{R}^p$ of the system at time t.

For example $p = 6$ *if* $\mathbf{y}$ *is a vector of orbital elements.*

$\boldsymbol{\mu} \in \mathbb{R}^{p'}$ are called dynamical parameters.

The integral flow, solution of (8) for initial data $\mathbf{y}_0$ at time t_0, is denoted by $\Phi_{t_0}^{t}(\mathbf{y}_0, \boldsymbol{\mu})$.
We also introduce the observation function

$$\mathbf{R} = (R_1, \ldots, R_k)\,, \qquad R_j = R_j(\mathbf{y}, t, \boldsymbol{\nu})\,, \quad j = 1 \ldots k$$

depending on the state $\mathbf{y}$ of the system at time t.

$\boldsymbol{\nu} \in \mathbb{R}^{p''}$ are called kinematical parameters.

Moreover we define the prediction function

$$\mathbf{r} = (r_1, \ldots, r_k), \qquad \mathbf{r}(t) = \mathbf{R}(\boldsymbol{\Phi}_{t_0}^{t}(\mathbf{y}_0, \boldsymbol{\mu}), t, \boldsymbol{\nu}).$$

The components r_i give a prediction for a specific observation at time t, e.g. the right ascension $\alpha(t)$, or the declination $\delta(t)$.

We can group the multidimensional data and predictions into two arrays, with components

$$r_i, \quad r(t_i), \qquad i = 1 \ldots m$$

respectively, and define the vector of the residuals

$$\boldsymbol{\xi} = (\xi_1, \ldots, \xi_m)\,, \qquad \xi_i = r_i - r(t_i)\,, \qquad i = 1 \ldots m\,.$$

The least squares method

The least squares principle asserts that *the solution of the orbit determination problem makes the target function*

$$\mathcal{Q}(\boldsymbol{\xi}) = \frac{1}{m}\, \boldsymbol{\xi}^T\, \boldsymbol{\xi} \tag{9}$$

attain its minimal value.

We observe that

$$\boldsymbol{\xi} = \boldsymbol{\xi}(\mathbf{y}_0, \boldsymbol{\mu}, \boldsymbol{\nu})$$

and select part of the components of $(\mathbf{y}_0, \boldsymbol{\mu}, \boldsymbol{\nu}) \in \mathbb{R}^{p+p'+p''}$ to form the vector $\mathbf{x} \in \mathbb{R}^N$ of the fit parameters, i.e. the parameters to be determined by fitting them to the data.

The least squares method

Let us define

$$Q(\mathbf{x}) = \mathcal{Q}(\boldsymbol{\xi}(\mathbf{x}; \mathbf{z})),$$

with $\mathbf{z}$ the vector of consider parameters, i.e. the remaining components of $(\mathbf{y}_0, \boldsymbol{\mu}, \boldsymbol{\nu})$ fixed at some assumed value.

An important requirement is that $m \geq N$.

We introduce the $m \times N$ design matrix

$$B = \frac{\partial \boldsymbol{\xi}}{\partial \mathbf{x}}(\mathbf{x})$$

and search for the minimum of $Q(\mathbf{x})$ by looking for solutions of

$$\frac{\partial Q}{\partial \mathbf{x}} = \frac{2}{m}\, \boldsymbol{\xi}^T B = \mathbf{0}\,. \tag{10}$$

To search for solutions of (10) we can use Newton's method.

The least squares method

Newton's method involves the computation of the second derivatives of the target function:

$$\frac{\partial^2 Q}{\partial \mathbf{x}^2} = \frac{2}{m}\, C_{new}, \qquad C_{new} = B^T B + \boldsymbol{\xi}^T H, \tag{11}$$

where

$$H = \frac{\partial^2 \boldsymbol{\xi}}{\partial \mathbf{x}^2}(\mathbf{x})$$

is a 3-index array of shape $m \times N \times N$.

By $\boldsymbol{\xi}^T H$ we mean the matrix with components $\sum_i \xi_i \dfrac{\partial^2 \xi_i}{\partial x_j \partial x_k}$.

Differential corrections

A variant of Newton's method, known as differential corrections, is often used to minimize the target function $Q(\mathbf{x})$.
We can take the normal matrix $C = B^T B$ in place of C_{new}.
At each iteration we have

$$\mathbf{x}_{k+1} = \mathbf{x}_k - C^{-1} B^T \xi$$

where B is computed at $\mathbf{x}_k$.

This approximation works if the residuals are small enough.

Let $\mathbf{x}_*$ be the value of $\mathbf{x}$ at convergence. The inverse of the normal matrix

$$\Gamma = C^{-1} \tag{12}$$

is called covariance matrix and its value in $\mathbf{x}_*$ can be used to estimate the uncertainty of the solution of the differential corrections algorithm.

Lecture II

Charlier's theory of multiple solutions and its generalization

Equations for preliminary orbits

From the geometry of the observations we have

$$r^2 = \rho^2 + 2q\rho\cos\epsilon + q^2 \qquad \text{(geometric equation).} \qquad (13)$$

From the two-body dynamics, both Laplace's and Gauss' method yield an equation of the form

$$\mathcal{C}\,\frac{\rho}{q} = \gamma - \frac{q^3}{r^3} \qquad \text{(dynamic equation)}\,, \qquad (14)$$

with $\mathcal{C}, \gamma$ real parameters depending on the observations.

Preliminary orbits and multiple solutions

intersection problem:

$$\begin{cases} (q\gamma - \mathcal{C}\rho)r^3 - q^4 = 0 \\ r^2 - q^2 - \rho^2 - 2q\rho\cos\epsilon = 0 \\ r, \rho > 0 \end{cases} \tag{15}$$

reduced problem:

$$P(r) = 0\,, \qquad r > 0 \tag{16}$$

with

$$P(r) = \mathcal{C}^2 r^8 - q^2(\mathcal{C}^2 + 2\mathcal{C}\gamma\cos\epsilon + \gamma^2)r^6 + 2q^5(\mathcal{C}\cos\epsilon + \gamma)r^3 - q^8.$$

We investigate the *existence of multiple solutions of the intersection problem*.

Charlier's theory

Carl V. L. Charlier (1862-1934)

In 1910 Charlier gave a geometric interpretation of the occurrence of multiple solutions in preliminary orbit determination with Laplace's method, assuming geocentric observations ($\gamma = 1$).

'the condition for the appearance of another solution simply depends on the position of the observed body' (MNRAS, 1910)

Charlier's hypothesis: $\mathcal{C}, \epsilon$ *are such that a solution of the corresponding intersection problem with* $\gamma = 1$ *always exists*.

A *spurious solution* of (16) is a positive root $\bar{r}$ of $P(r)$ that is not a component of a solution $(\bar{r}, \bar{\rho})$ of (15) for any $\bar{\rho} > 0$.

We have:

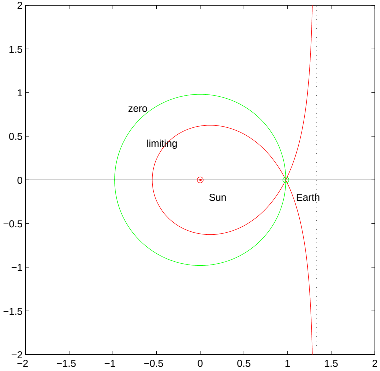
- $P(q) = 0$, and $r = q$ corresponds to the observer position;
- $P(r)$ has always 3 positive and 1 negative real roots.

Let $P(r) = (r - q)P_1(r)$: then

$$P_1(q) = 2q^7 \mathcal{C}[\mathcal{C} - 3\cos\epsilon].$$

If $P_1(q) < 0$ there are 2 roots $r_1 < q$, $r_2 > q$; one of them is spurious.
If $P_1(q) > 0$ both roots are either $< q$ or $> q$; they give us 2 different solutions of (15).

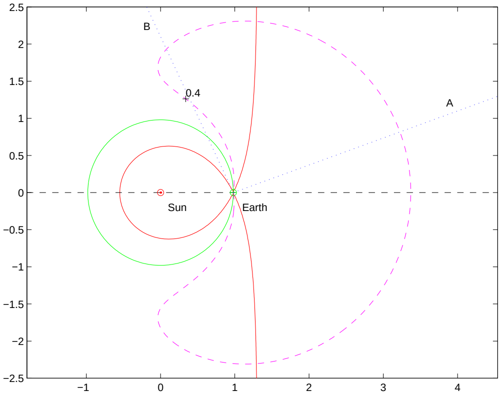
zero circle: $\mathcal{C} = 0$,
limiting curve: $\mathcal{C} - 3\cos\epsilon = 0$.



The green curve is the zero circle. The red curve is the limiting curve, whose equation in heliocentric rectangular coordinates (x, y) is

$$4 - 3\frac{x}{q} = \frac{q^3}{r^3}.$$

Geometry of the solutions



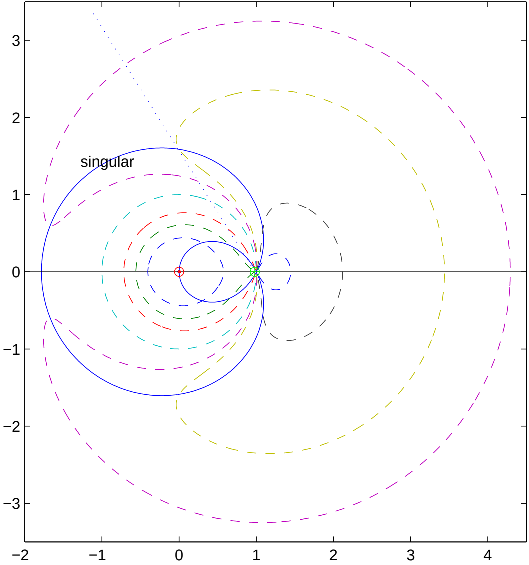
The position of the observed body corresponds to the intersection of the level curve $\mathcal{C}^{(1)}(x,y) = \mathcal{C}$ with the observation line (defined by ϵ), where $\mathcal{C}^{(1)} = \mathcal{C}^{(1)} \circ \Psi$,

$$\mathcal{C}^{(1)}(r,\rho) = \frac{q}{\rho}\left[1 - \frac{q^3}{r^3}\right]$$

and $(x,y) \mapsto \Psi(x,y) = (r,\rho)$ is the map from rectangular to bipolar coordinates.

Note that the position of the observed body defines an intersection problem.

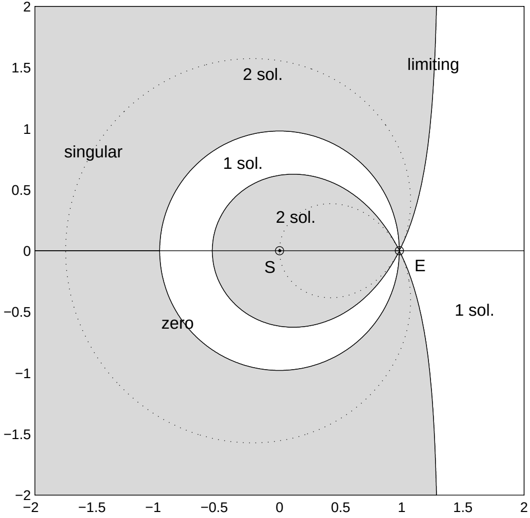
The singular curve



The singular curve is the set of tangency points between an observation line and a level curve of $\mathcal{C}^{(1)}$. It can be defined by

$$4 - 3q\frac{x}{r^2} = \frac{r^3}{q^3}.$$

Multiple solutions: summary



Alternative solutions occurs in 2 regions: the interior of the limiting curve loop and outside the zero circle, on the left of the unbounded branches of the limiting curve.

See Gronchi, G.F.: CMDA **103/4** (2009)

Let $\gamma \in \mathbb{R}, \gamma \neq 1$. By the dynamic equation we define

$$\mathfrak{C}^{(\gamma)} = \mathcal{C}^{(\gamma)} \circ \Psi \,, \qquad \mathcal{C}^{(\gamma)}(r,\rho) = \frac{q}{\rho}\left[\gamma - \frac{q^3}{r^3}\right]$$

with $(x,y) \mapsto \Psi(x,y) = (r,\rho)$.

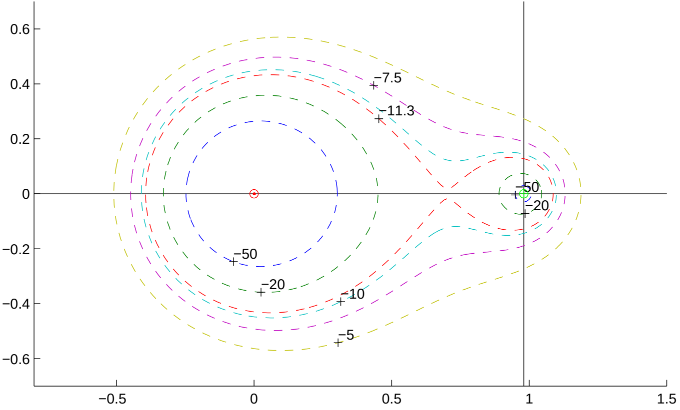
We also define the zero circle, with radius

$$r_0 = q/\sqrt[3]{\gamma}, \qquad \text{for } \gamma > 0.$$

Introduce the following assumption:

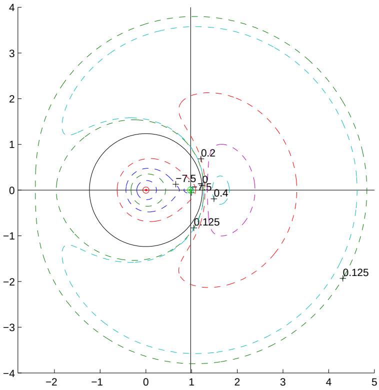
the parameters $\gamma, \mathcal{C}, \epsilon$ are such that the corresponding intersection problem admits at least one solution. (17)

Topology of the level curves of $\mathcal{C}^{(\gamma)}$

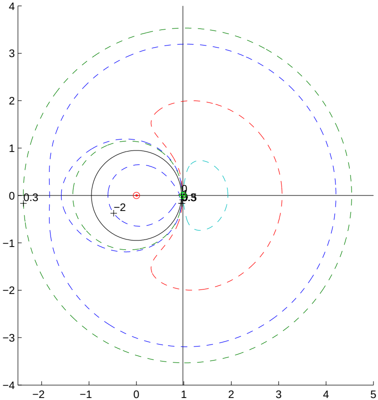


$\gamma \leq 0$

Topology of the level curves of $\mathcal{C}^{(\gamma)}$



$0 < \gamma < 1$

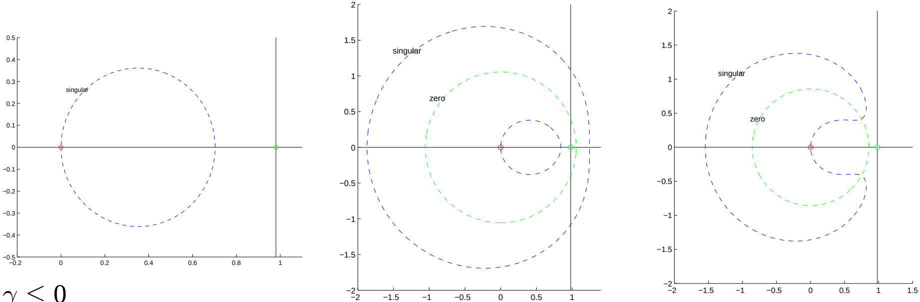


$\gamma > 1$

The singular curve

For $\gamma \neq 1$ we *cannot* define the limiting curve by Charlier's approach, in fact $P(q) \neq 0$. Nevertheless we can define the singular curve as the set

$$\mathcal{S} = \{(x,y) : \mathcal{G}(x,y) = 0\}\,, \qquad \mathcal{G}(x,y) = -\gamma r^5 + q^3(4r^2 - 3qx).$$



$\gamma \leq 0$ $\qquad$ $0 < \gamma < 1$ $\qquad$ $\gamma > 1$

An even or an odd number of solutions

The solutions of an intersection problem (15) can not be more than 3. In particular, for $(\gamma, \mathcal{C}, \epsilon)$ fulfilling (17) with $\gamma \neq 1$,
if the number of solutions is even they are 2,
if it is odd they are either 1 or 3.
For $\gamma \neq 1$ we define the sets

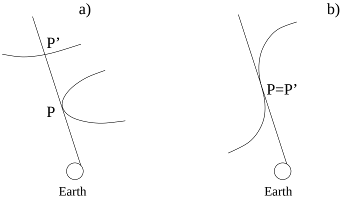
$$\mathcal{D}_2(\gamma) = \begin{cases} \emptyset & \text{if } \gamma \leq 0 \\ \{(x,y) : r > r_0\} & \text{if } 0 < \gamma < 1 \\ \{(x,y) : r \leq r_0\} & \text{if } \gamma > 1 \end{cases}$$

and

$$\mathcal{D}(\gamma) = \mathbb{R}^2 \setminus (\mathcal{D}_2(\gamma) \cup \{(q,0)\}) \,.$$

Points in $\mathcal{D}_2(\gamma)$ corresponds to intersection problems with 2 solutions; points in $\mathcal{D}(\gamma)$ to problems with 1 or 3 solutions.

Residual points



Fix $\gamma \neq 1$ and let $(\bar{\rho}, \overline{\psi})$ correspond to a point $\mathrm{P} \in \mathfrak{S} = \mathcal{S} \bigcap \mathcal{D}$. Let

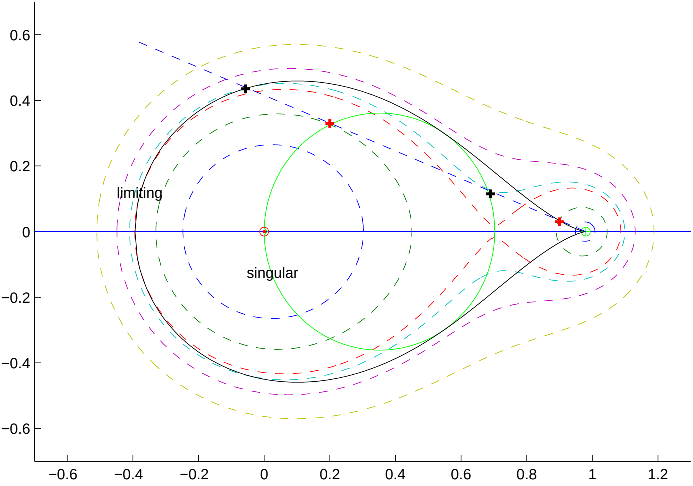
$$F(\mathcal{C}, \rho, \psi) = \mathcal{C}\frac{\rho}{q} - \gamma + \frac{q^3}{r^3},$$

If $F_{\rho\rho}(\mathcal{C}, \bar{\rho}, \overline{\psi}) \neq 0$, we call residual point related to P the point $\mathrm{P}' \neq \mathrm{P}$ lying on the same observation line and the same level curve of $\mathcal{C}^{(\gamma)}(x, y)$, see Figure a).

If $F_{\rho\rho}(\mathcal{C}, \bar{\rho}, \overline{\psi}) = 0$ we call P a self–residual point, see Figure b).

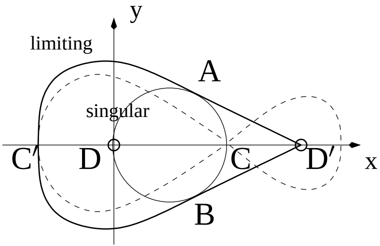
The limiting curve

Let $\gamma \neq 1$. The limiting curve is the set composed by all the residual points related to the points in $\mathfrak{S}$.



The limiting curve

Separating property: *the limiting curve $\mathcal{L}$ separates $\mathcal{D}$ into two connected regions $\mathcal{D}_1, \mathcal{D}_3$: $\mathcal{D}_3$ contains the whole portion $\mathfrak{S}$ of the singular curve. If $\gamma < 1$ then $\mathcal{L}$ is a closed curve, if $\gamma > 1$ it is unbounded.*



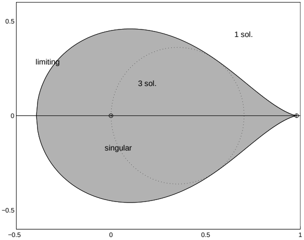
$(\gamma \leq 0)$

The limiting curve

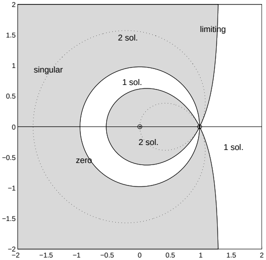
Transversality: *the level curves of* $\mathcal{C}^{(\gamma)}(x,y)$ *cross* $\mathcal{L}$ *transversely, except for at most the two self–residual points and for the points where* $\mathcal{L}$ *meets the* x*-axis.*

Limiting property: For $\gamma \neq 1$ the limiting curve $\mathcal{L}$ divides the set $\mathcal{D}$ into two connected regions $\mathcal{D}_1, \mathcal{D}_3$: the points of $\mathcal{D}_1$ are the unique solutions of the corresponding intersection problem; the points of $\mathcal{D}_3$ are solutions of an intersection problem with three solutions.

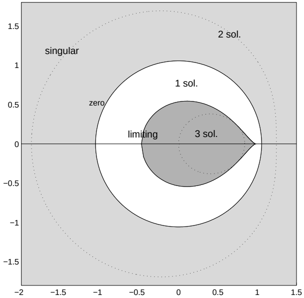
Multiple solutions: the big picture



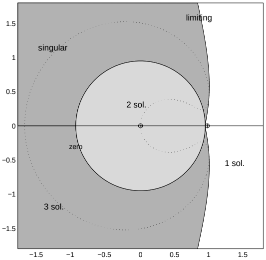
$\gamma \leq 0$



$\gamma = 1$



$0 < \gamma < 1$



$\gamma > 1$