Methods for Optimizing Interplanetary Trajectories

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Motivation (1)

- There are several different methods already proposed for NEA mitigation including:
  - (sudden) deflection by collision
  - (sudden) deflection by nuclear explosion
  - (slow) deflection using gravity tractor
  - (slow) deflection using Yarkovsky effect
  - (slow) deflection using surface mass driver
- These different methods can require different types of spacecraft trajectories, e.g. the first requires interception while the others need rendezvous.
- The trajectories in either case may use impulsive (chemical) thrust, or low (electric) thrust, or a combination of the two.
- The trajectories may make use of planetary flybys (gravity assist maneuvers)

Motivation (2)

•These trajectories are all too challenging to solve analytically – need a numerical solution.

•Unfortunately each combination of possible trajectory elements:

interception/rendezvous

impulsive thrust/ low-thrust/ combination

planetary flybys/ no planetary flybys

may require a different numerical solution approach.

•However one thing that nearly all such solutions will have in common, based on our experience, is that accurate solutions require a two-step process:

Find approximate solution (guess) Þ Refine guess with accurate solver

Motivation (3)

- So this series of three lectures is intended to (only):
  - Show how to formulate the trajectory optimization problem for some of the mitigation strategies
  - Describe the numerical solution methods available
  - Show what numerical solution approaches have been successful with a number of these possible missions (via examples)

(Note that the first two items alone could easily constitute a year-long graduate-level class.)

Lecture 1: Methods for Optimizing Interplanetary Trajectories I

The problem: objectives, equations of motion, typical boundary conditions

Methods of solution, indirect vs. direct

Brief Summary of Extant Methods for Trajectory Optimization

- 1960's - Hamiltonian methods, primer vector theory
- 1980's - Direct transcription (DT), collocation with NLP
- 2000's - Evolutionary algorithms (EA) and metaheuristics

Most important distinction: impulsive or low-thrust propulsion – impulsive case is a parameter optimization problem/ L-T case is a continuous optimization problem

Low-thrust case much more challenging. Possible solution strategies:

- Hamiltonian (COV) methods
- direct transcription (DT)
- "shape-based" methods
- Sims-Flanagan transcription

The need for an initial guess of the solution and how to obtain one

Lecture 2: Methods for Optimizing Interplanetary Trajectories II / Application to the Problem of NEA Deflection

Metaheuristic methods for solution of optimal spacecraft trajectory problems: as initial guess for a more precise method *or* as solution in own right

Need different approaches for “naturally discrete” problems/ continuous-thrust problems

Examples

Proposed methods of asteroid deflection: collision, nuclear explosion, mass drivers, gravity tractor. Advantages and disadvantages.

The objective: maximizing deflection (easy) vs. maximizing displacement from Earth surface (complicated)

Using the state transition matrix to optimize deflection via collision

Lecture 3: Asteroid Deflection/ Characterization by Sample Return

Optimizing asteroid deflection via collision using a low-thrust spacecraft

Setting up the problem for solution by a numerical optimization method

Example trajectories and resulting deflections

A related problem: reconnaissance of a potentially dangerous asteroid

Optimizing a reconnaissance/sample return mission to an asteroid using particle swarm optimization

Examples and results

The Spacecraft Trajectory Optimization Problem

Objective: minimize propellant mass required, minimize flight time,
maximize miss distance of NEA at time of closest approach

Equations of Motion: Can be directly based on Newton's 2nd law, e.g.

$$\dot{\bar{x}} = \bar{f} = \begin{bmatrix} \dot{\bar{r}} \\ \dot{\bar{v}} \end{bmatrix} = \begin{bmatrix} \bar{v} \\ \bar{g}(\bar{r}) + G\hat{u} \end{bmatrix}$$

or, may be Gauss or Lagrange form of variational eqns.
in terms of orbital elements, i.e.

$$\dot{\bar{x}} = \begin{bmatrix} \dot{a} \\ \dot{e} \\ \dot{i} \\ \dot{W} \\ \dot{w} \\ \dot{M} \\ \dot{m} \end{bmatrix} = \bar{f}$$

The Spacecraft Trajectory Optimization Problem

Typical boundary conditions: arrive into specified orbit at destination planet (Galileo)
perform flyby of destination planet (Voyager)
arrive into halo orbit (ISEE-3)
land on planetary surface
orbit transfer (e.g. LEO to GEO)

Methods of Solution: Indirect vs. Direct

Solutions can be categorized into two basic types: Indirect and Direct

<u>Indirect</u>

Uses the analytical necessary conditions (NC) of the calculus of variations (COV)

These are sometimes also referred to as *Hamiltonian* methods

A particular application of the COV to space trajectory optimization yields "primer vector" theory

Few analytical solutions possible; must solve the NC using a numerical method

Characterized by having few, 10's to 100's, of free parameters (usually initial multipliers)

<u>Direct</u>

Transforms the continuous optimization problem into a discrete parameter optimization (NLP) problem

Many transcription methods extant: direct collocation, Gauss pseudospectral methods, R-K parallel shooting

Very simply incorporates control bounds and boundary conditions (terminal and path constraints)

Typically yields 100's to 1000's of free parameters (usually states and controls)

The Optimal Control Problem

System: $\dot{x} = f(x,\ u,\ t),\ \ x(0)\ given$

$$\psi\left[x(T),\ T\right] = 0,\ \ q\ constraint\ eqns.$$

Problem: Find control u(t) to minimize $J = \phi\left[x(T),\ T\right] + \int_0^T L\left[x,\ u,\ t\right] dt$

Calculus of Variations necessary conditions: (Euler-Lagrange equations)

$$Define\ Hamiltonian\ H = L + \lambda^T f,\ then$$

$$\dot{\lambda} = -\left(\frac{\partial H}{\partial x}\right)^T,\quad \lambda(T) = \left[\left(\frac{\partial \phi}{\partial x}\right) + \nu^T \frac{\partial \psi}{\partial x}\right]^T_{t=T}$$

$$\frac{\partial H}{\partial u} = 0,$$

$$either\ T\ given,\ or$$

$$\left[\frac{\partial \phi}{\partial t} + \nu^T \frac{\partial \psi}{\partial t} + \left(\frac{\partial \phi}{\partial x} + \nu^T \frac{\partial \psi}{\partial x}\right) f + L\right]_{t=T} = 0$$

Analytical (COV-Based) Methods

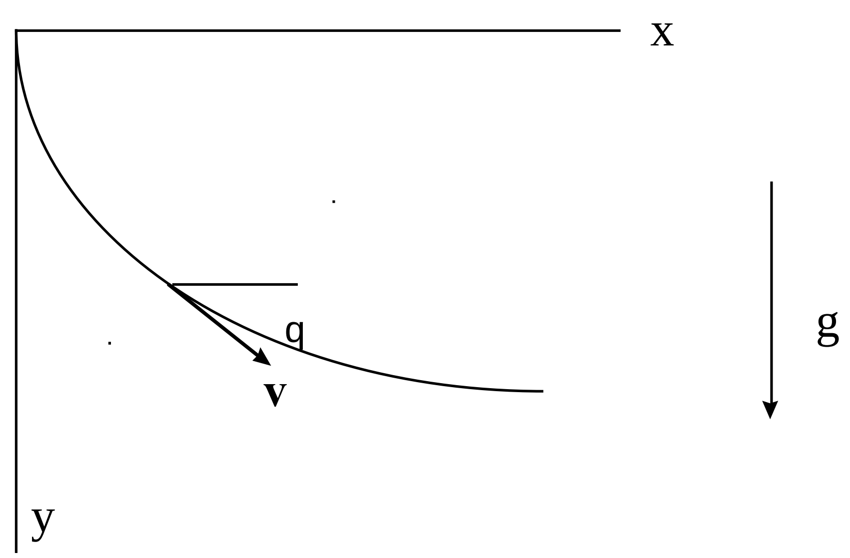
Example
the Brachistochrone (1696)

$$\dot{x} = \sqrt{2gy}\cos\theta,\ x(0) = 0$$

$$\dot{y} = \sqrt{2gy}\sin\theta, y(0) = y_0,\ y(t_f) = y_0$$

Define

$$H = 1 + \lambda_x\sqrt{2gy}\cos\theta + \lambda_y\sqrt{2gy}\sin\theta$$



Then, the necessary conditions for
an optimal trajectory become:

$$\dot{\lambda}_x = -\frac{\partial H}{\partial x} = 0$$

$$\dot{\lambda}_y = -\frac{\partial H}{\partial y} = -\sqrt{\frac{g}{2y}}\lambda_x\cos\theta - \sqrt{\frac{g}{2y}}\lambda_y\sin\theta$$

with

$$\lambda_x(t_f) = 0$$

$$\lambda_y(t_f) = \nu_y \text{ , a constant}$$

and

$$\frac{\partial H}{\partial \theta} = 0 = -\lambda_x\sin\theta + \lambda_y\cos\theta \Rightarrow \frac{\lambda_y}{\lambda_x} = \tan\theta$$

with $\mathrm{H}(t_f) = 0$

This is a TPBVP with 4 ODE's. The optimal control is determined directly by the Lagrange multipliers through the optimality condition.

Continuous System with Control Constraints

When there are constraints on the magnitude of the control u the control is determined according to:

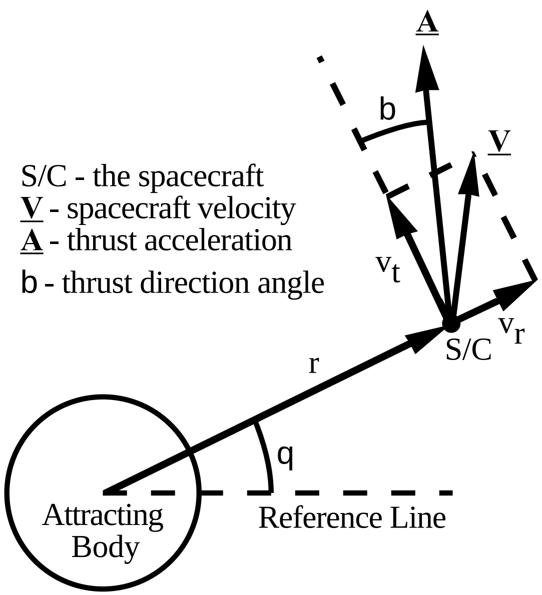
$$\frac{\partial H}{\partial u} = 0 \quad \textit{if} \text{ the resulting } u \text{ satisfies the constraints,}$$

or, u is chosen to minimize the instantaneous value of the Hamiltonian H (Pontryagin minimum principle).

In both cases, the optimality condition is a function of the instantaneous states, the instantaneous values of the Lagrange multipliers (costates), or both.

Lagrange and Me

Generic Spacecraft in Inverse-Square Gravitational Field



- The system differential equations become

$$\frac{dr}{dt} = v_r$$

$$\frac{d\theta}{dt} = v_\theta / r$$

$$\frac{dv_r}{dt} = v_\theta^2/r + g(r) + s\,a\sin\beta$$

$$\frac{dv_\theta}{dt} = -v_r v_\theta/r + s\,a\cos\beta$$

$$\frac{da}{dt} = s\,a^2/c.$$

$$or \;\; \bar{x} = \left[r,\ \theta,\ v_r,\ v_\theta,\ a\right]^T, \;\; u = \beta$$

where: s is the throttling parameter, $0 \le s \le 1$

gravitational force function $g(r) = -1/r^2$

Necessary Conditions – Lagrange Multiplier DE's

In this coordinate system the multiplier (costate) differential equations become:

$$\frac{d\lambda}{dt} = -F(x,u)^T \lambda, \text{ with } F(x,u) = \frac{\partial f}{\partial x}$$

where $\lambda = \left[\lambda_r, \lambda_\theta, \lambda_{vr}, \lambda_{v\theta}, \lambda_a\right]^{\mathrm{T}}$, $\mathrm{x}=\left[\mathrm{r}, \theta, v_r, v_\theta, a\right]^T$, $u = \beta$, and the F matrix is:

$$F = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ -v_\theta / r^2 & 0 & 0 & 1/r & 0 \\ -v_\theta^2/r^2 + 2/r^3 & 0 & 0 & 2v_\theta / r & s\sin\beta \\ v_r v_\theta / r^2 & 0 & -v_\theta / r & -v_r / r & s\cos\beta \\ 0 & 0 & 0 & 0 & 2sa/c \end{bmatrix}$$

Possible Objectives

For the <u>impulsive case</u> the fuel-minimizing trajectory minimizes the sum of the impulses, i.e.

$$J = \sum_{i=1}^{N} \left| \Delta \bar{V}_i \right|$$

For the <u>continuous thrust</u> case the fuel-minimizing trajectory minimizes:

$$J = \int_0^{t_f} \sigma a \; dt = \int_0^{t_f} \sigma \frac{T}{m} \; dt$$

where T is the thrust provided by the engine and σ is the throttling parameter ($0 \le \sigma \le 1$)

For the <u>minimum-time</u> case the objective becomes

$$\mathrm{J} = \phi\left[x(t_f), t_f \right] = t_f \quad or \quad J = \int_{t_0}^{t_f} L \; dt = \int_{t_0}^{t_f} 1 \; dt$$

Necessary Conditions - Optimal Control

The Hamiltonian can be written in a more compact form by

defining: $\bar{r} = \begin{bmatrix} r \\ \theta \end{bmatrix}$, $\bar{v} = \begin{bmatrix} v_r \\ v_\theta \end{bmatrix}$, $\bar{\lambda}_r = \begin{bmatrix} \lambda_r \\ \lambda_\theta \end{bmatrix}$, $\bar{\lambda}_v = \begin{bmatrix} \lambda_{vr} \\ \lambda_{v\theta} \end{bmatrix}$, $\bar{u} = \begin{bmatrix} \cos\beta \\ \sin\beta \end{bmatrix}$

Consider the minimum-fuel case with $L = \sigma$, then

$$H = \sigma + \bar{\lambda}_r^T \bar{v} + \bar{\lambda}_v^T \left[\bar{g}(\bar{r}) + \sigma\alpha\bar{u} \right] + \lambda_\alpha \sigma \alpha^2 / c$$

$$or, \quad H = \sigma\left(1 + \alpha\bar{\lambda}_v^T \bar{u} + \lambda_\alpha \alpha^2 / c\right) + \bar{\lambda}_r^T \bar{v} + \bar{\lambda}_v^T \bar{g}(\bar{r})$$

to minimize H over all possible $\bar{u}$, one chooses

$\bar{u} = -\bar{\lambda}_v$ "primer vector theory" (1960's) (which is equivalent to $\tan\beta = \dfrac{\lambda_{vr}}{\lambda_{v\theta}}$)

Necessary Conditions-Switching Function

With $\bar{u} = -\bar{\lambda}_v$, the Hamiltonian becomes:

$$H = s\left(1 - a\lambda_v + \lambda_a a^2 / c\right) + \bar{\lambda}_r^T \bar{\mathrm{v}} + \bar{\lambda}_v^T \bar{g}(\bar{r})$$

Let $S = \left(1 - a\lambda_v + \lambda_a a^2 / c\right)$

To minimize H over the throttling parameter s one chooses:

$s = 1$ if $S > 0$

$s = 0$ if $S < 0$

i.e. $S = \left(1 - a\lambda_v + \lambda_a a^2 / c\right)$ is the "switching function" for the control.

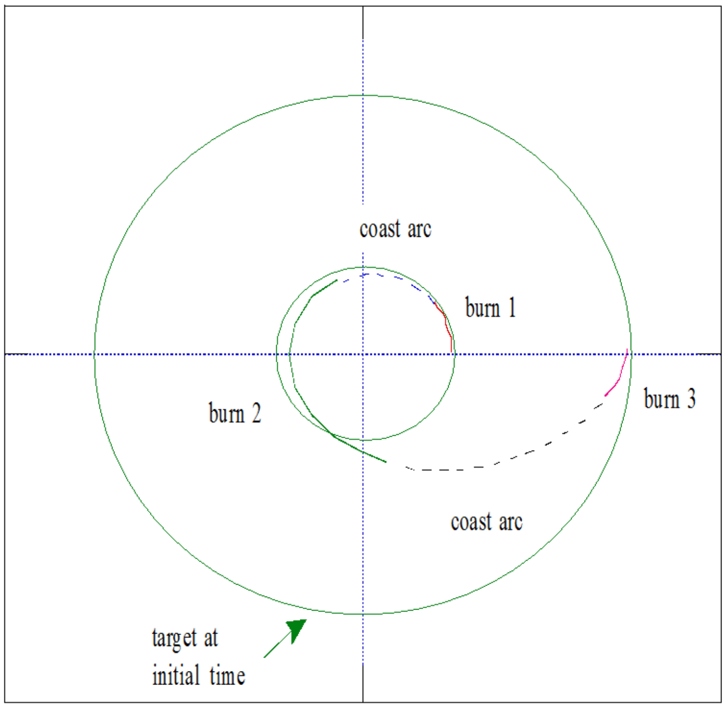
The optimal control may consist of a sequence of alternating max.

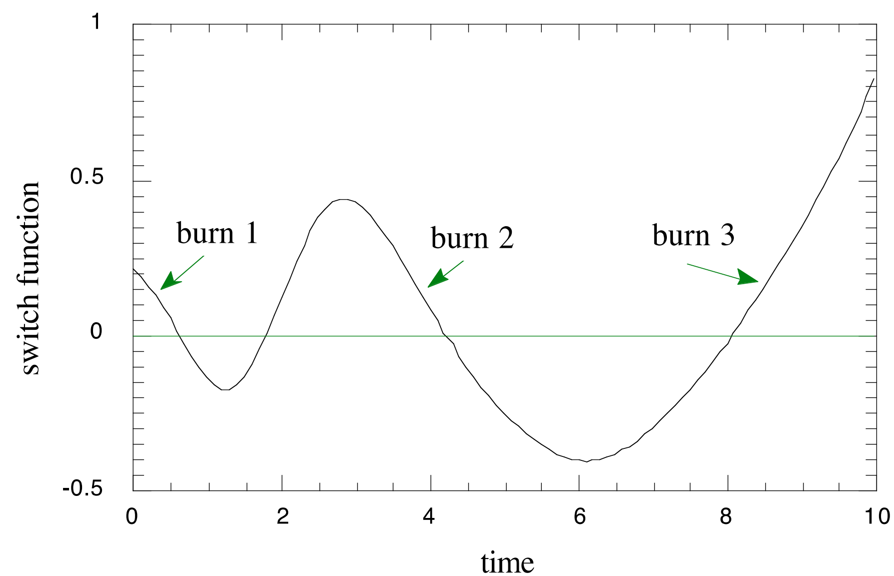
thrust arcs and coast arcs.

The problem is solved if the initial values of the Lagrange multipliers can be determined!

Example: Optimal (minimum-fuel) Low-Thrust Rendezvous, Solved Using the Analytical Necessary Conditions

- $r_{init} = 1$, $r_{final} = 3$, target lead angle 4.5 rad, $a_0 = 0.1$, $c = 1.5$
- Initial and final orbits are circular
- A TPBVP solver is used to find the required initial values of $\lambda_r, \lambda_\theta, \lambda_{v_r}, \lambda_{v_\theta}, \lambda_a$





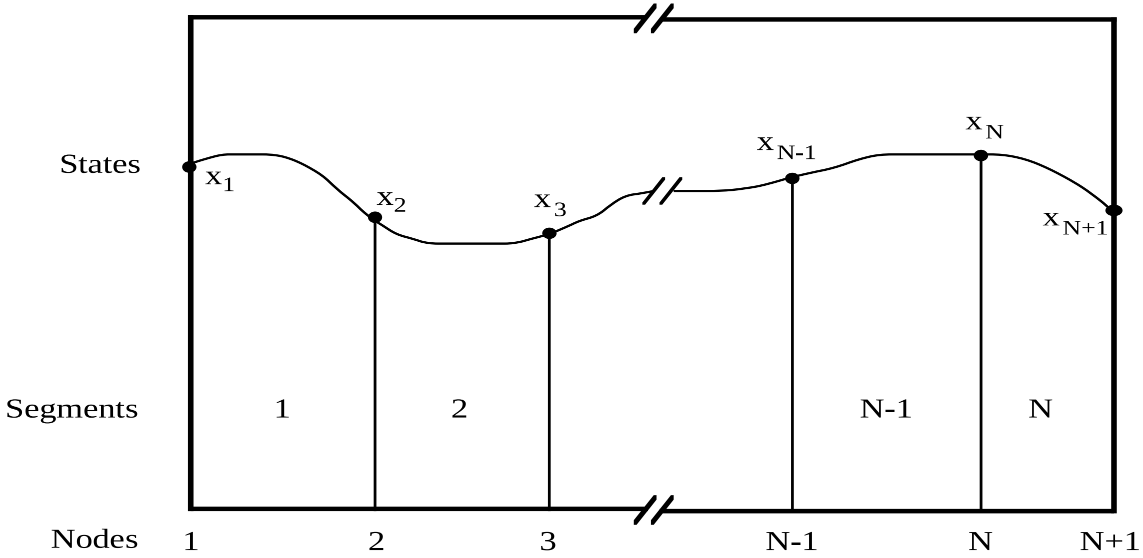
Analytical (COV-Based) Methods   Advantages and Disadvantages

- Advantages
  - Familiar
  - A solution provides information, via the *primer vector*, about how it may be improved, e.g. with a midcourse impulse or coast arc.
  - No question about optimality (local minimum)
  - Optimal control determined analytically
  - Costates represent sensitivity of cost to changes in the states

- Disadvantages
  - Immediately increases the dimension of the system by 2x
  - Can't cope with tabular data, e.g. thrust vs. altitude and temperature
  - Yields a TPBVP; difficult to solve without good initial states/costates
  - Changes in terminal conditions or constraints require changes in the system equations
  - Including path constraints (e.g. dynamic pressure constraints) is problematic

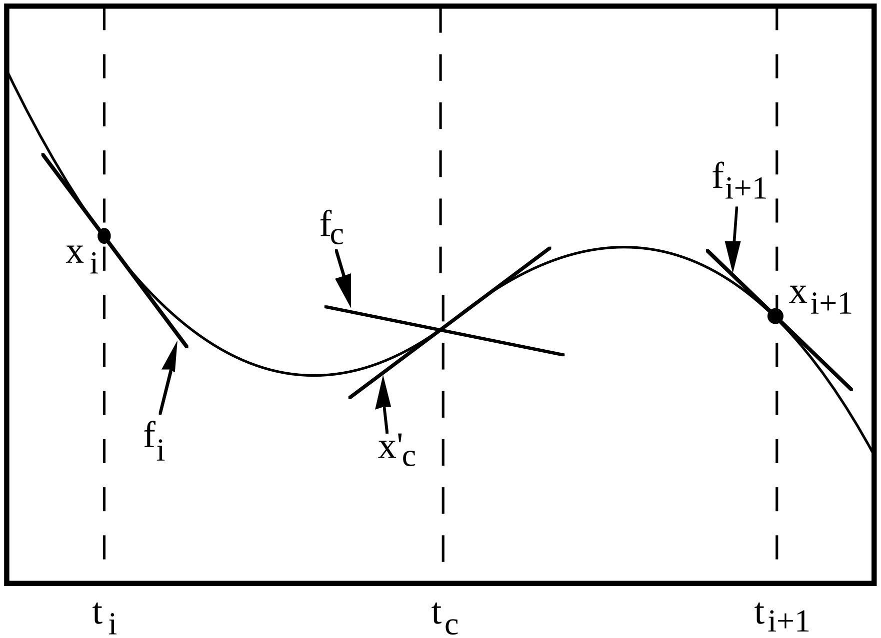
Direct Transcription (DT) Solutions

Collocation or *Pseudospectral* methods divide continuous history into segments.
States and controls are known only at discrete points.
Implicit integration rules, written as nonlinear constraints, enforce satisfaction of the equations of motion.
The optimal control problem is converted into a NLP problem.



Direct Transcription (DT) Solutions

In the *collocation* method we assume that within each segment the state is described by a polynomial in time, e.g. $x = c + c_1 s + c_2 s^2 + c_3 s^3$



Choose "collocation" point at the center of the segment. Then,

$$x(t_c) = x(s{=}1/2) = c_0 + \frac{c_1}{2} + \frac{c_2}{4} + \frac{c_3}{8}$$

$$= \frac{x_i + x_{i+1}}{2} + \frac{\Delta t_i \left[f(t_i) - f(t_{i+1}) \right]}{8}$$

$$\dot{x}(t_c) = \dot{x}(s{=}1/2) = \left[c_1 + c_2 + \frac{3c_3}{4} \right] \frac{1}{\Delta t_i}$$

$$= \frac{-3\left[x_i - x_{i+1} \right]}{2\Delta t_i} - \frac{\left[f(t_i) + f(t_{i+1}) \right]}{4}$$

Forcing $\dot{x}(t_c)$, the slope of the polynomial, to equal the function, evaluated at the center point, $f(t_c, x(t_c))$, yields

$$x_{i+1} - x_i - \frac{\Delta t_i}{6}\left[f(t_i) + 4\, f(t_c) + f(t_{i+1}) \right] = 0$$

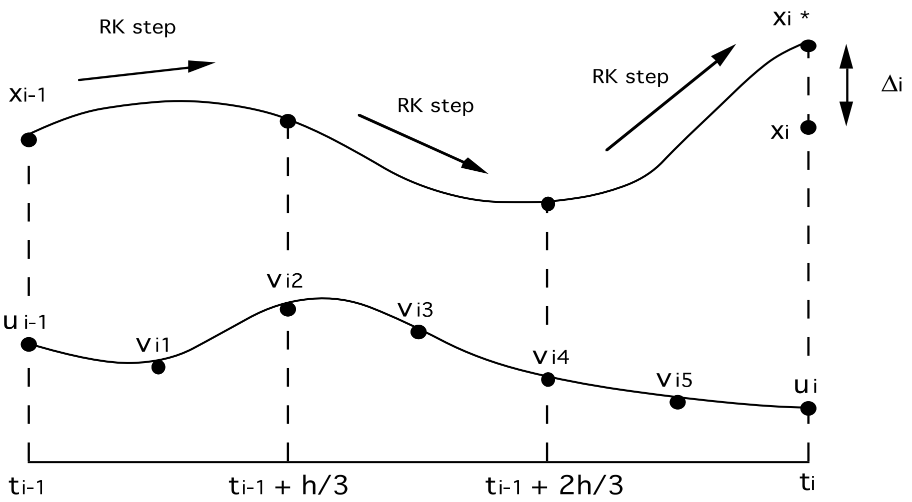
which is Simpson's rule!

Another DT scheme is *Runge-Kutta parallel-shooting*. R-K parallel shooting uses explicit numerical integration in creating the nonlinear constraints.

$$y_{i1}^1 = x_{i-1} + \frac{1}{2p} hf\left(x_{i-1}, u_{i-1}\right)$$

$$y_{i1}^2 = x_{i-1} + \frac{1}{2p} hf\left(y_{i1}^1, u_{i1}\right)$$

$$y_{i1}^3 = x_{i-1} + \frac{1}{p} hf\left(y_{i1}^2, u_{i1}\right)$$



$$y_{i1}^4 = x_{i-1} + \frac{1}{6p} h\left[f\left(x_{i-1}, u_{i-1}\right) + 2f\left(y_{i1}^1, u_{i1}\right) + 2f\left(y_{i1}^2, u_{i1}\right) + f\left(y_{i1}^3, u_{i2}\right)\right]$$

p = 3

$$D_i = y_{i3}^4 - x_i = 0$$

Direct Transcription (DT) Solutions   Conversion into NLP problem

Collect all independent variables into a single vector $\mathbf{P}^T = \left[\mathbf{Z}^T,\ \mathbf{E}^T\right]$

where $\mathbf{Z}^T = (\mathbf{x}_1^T,\ \mathbf{u}_1^T, \mathbf{x}_2^T,\ \mathbf{u}_2^T, \ldots,\ \mathbf{x}_{N+1}^T,\ \mathbf{u}_{N+1}^T)$

$$\mathbf{E}^T = (E_1,\ E_2, \ldots,\ E_L)$$

Problem is then of form: minimize $f\ (\mathbf{P})$ subject to constraints

$$\mathbf{b}_L \le \begin{Bmatrix} \mathbf{P} \\ A\mathbf{P} \\ \mathbf{C}(\mathbf{P}) \end{Bmatrix} \le \mathbf{b}_U$$

where A**P** contains all of the linear constraints, **C(P)** is a vector of all of the nonlinear constraints.

There will be (N + 1)(number of state variables + number of control variables) parameters in the vector **Z**; usually only a small number of parameters, such as switching times for motor operation, in the event vector **E**.

Direct Transcription (DT) Solutions Advantages and Disadvantages

- Advantages

  Straightforward to code

  Don't need to know optimal control theory; don't need possibly difficult analytical differentiation

  Generally robust; tolerant of poor initial guess

  Control constraints are included trivially

  Constraints such as dynamic pressure constraints, which are problematic for COV methods, are simply included

  Changes in terminal conditions or constraints easily made

- Disadvantages

  States and controls known only at discrete points

  No guarantee of optimality

  Need a sufficiently good initial guess – can sometimes use intuition or experience; for challenging problem probably need approximate numerical solution (or homotopy)

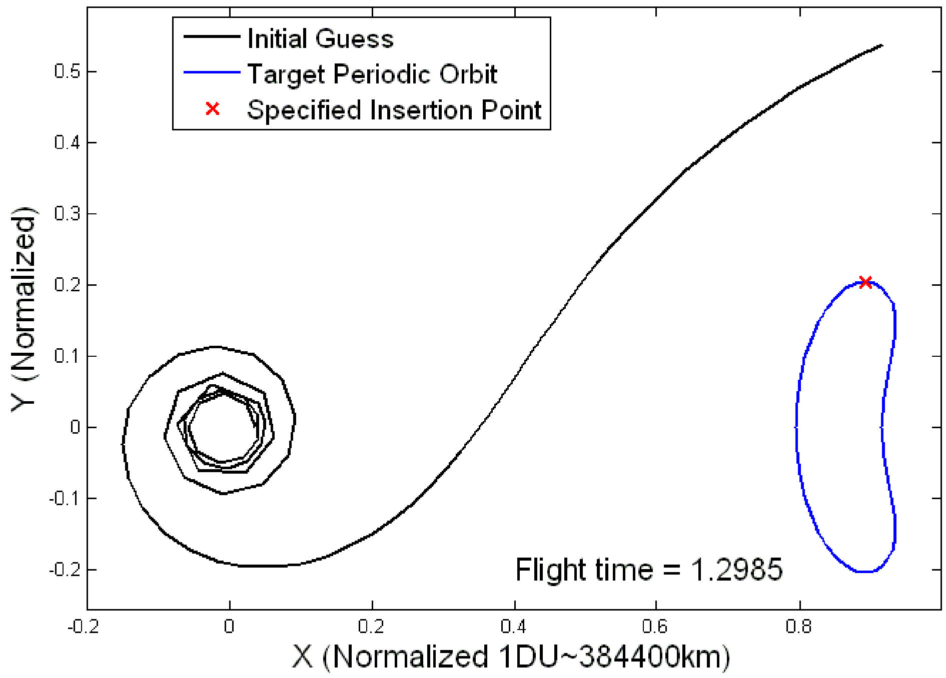
  Likely to converge to a minimum in the neighborhood of the initial guess

  The solution provides no information about possible improvement; it is the best solution for the given structure.

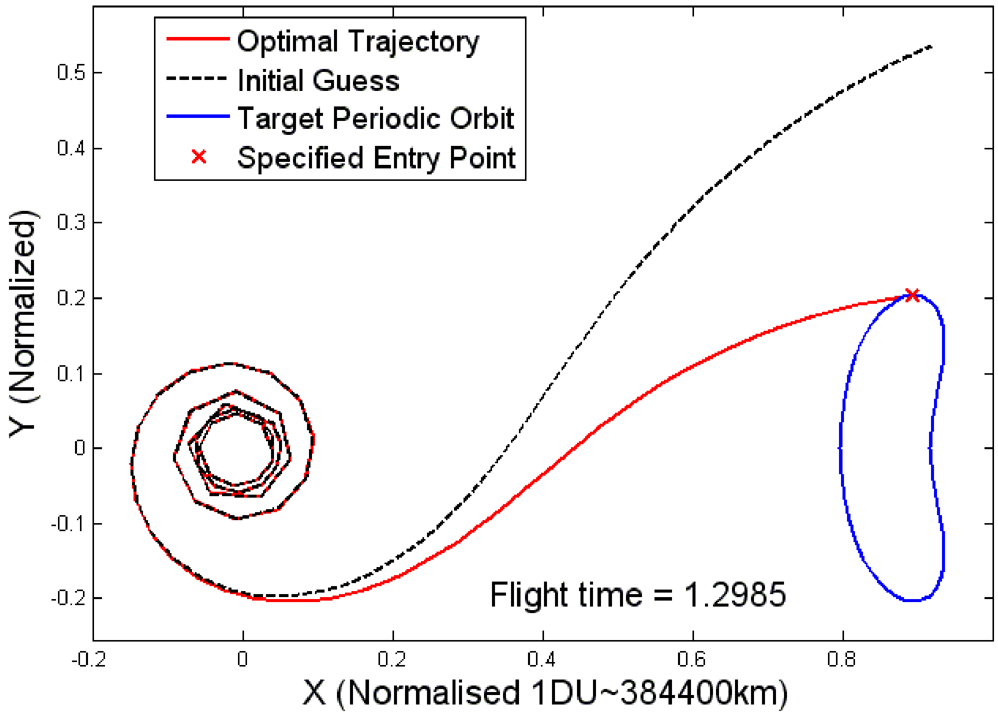
Example: Optimal Low-Thrust Transfer from LEO to Periodic Orbit About L_1, Solved Using Direct Transcription

The solution uses the collocation method with NLP

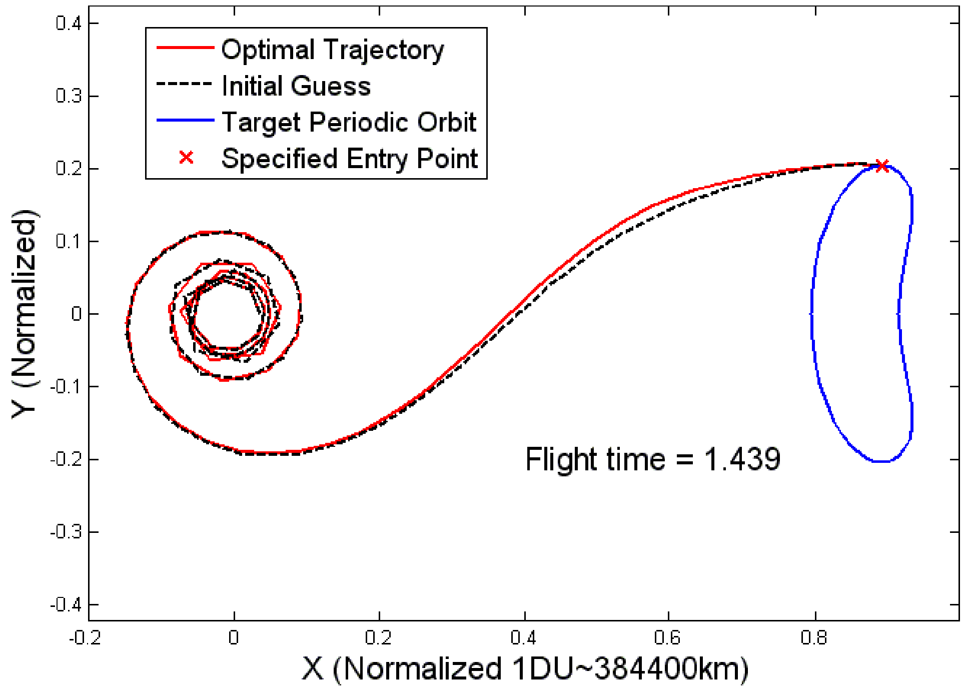
Step 1: The system DE's are numerically integrated assuming thrust is always directed along the velocity vector. The integration ends when the S/C is near the periodic orbit. This provides the initial guess.



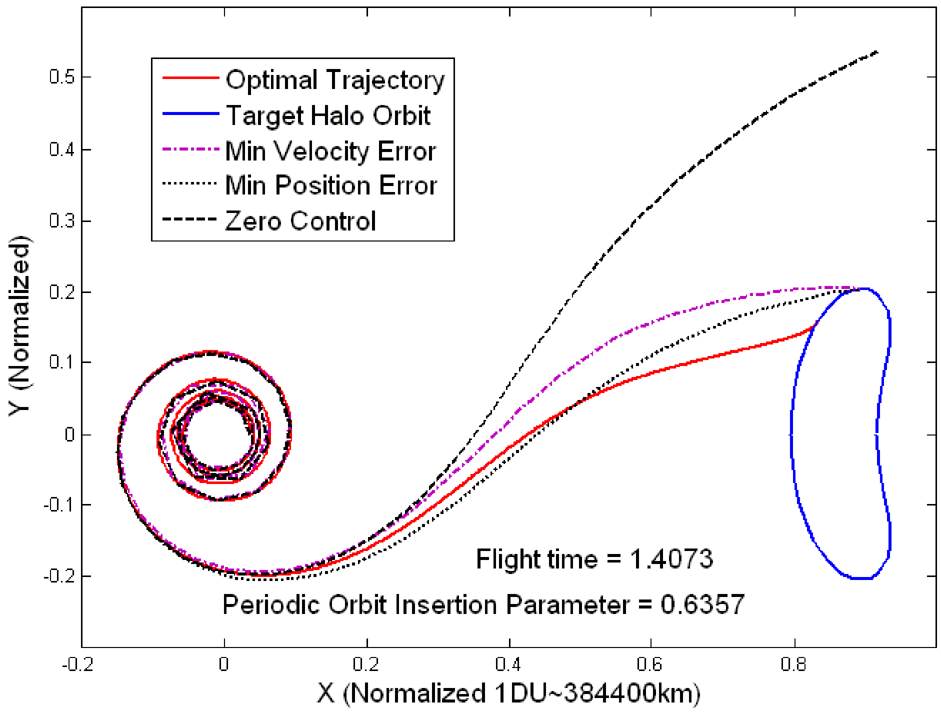
Step 2: The numerically integrated trajectory is used as an initial guess for the NLP solver. The optimizer is required to reach only a specific point on the periodic orbit, i.e. to minimize $f = (x_f - x_s)^2 + (y_f - y_s)^2$



Step 3: The previous trajectory is used as an initial guess for the NLP solver. The optimizer is required to achieve a specific velocity on the periodic orbit i.e. by minimizing $f = \left(v_{x_f} - v_{x_s}\right)^2 + \left(v_{y_f} - v_{y_s}\right)^2$ but with final position constrained to the same point (x_s, y_s).



Step 4: The previous solution is used as an initial guess for the NLP solver. The optimizer is now allowed to move the entry point to any position on the periodic orbit in order to minimize the final time, t_f, (which simultaneously minimizes fuel consumption.)



Evolutionary Algorithms (EA's) and Metaheuristics

"Evolutionary computation has as its objective to mimic processes from natural evolution, where the main concept is survival of the fittest: the weak must die."
A. Engelbrecht, *Computational Intelligence* (2007)

Among the best known and most often employed EA's and heuristics are:

Genetic Algorithms (GA) which model genetic evolution

Differential Evolution Algorithms (DE) similar to GA but for continuous-valued problems; also the mutation operator is dependent on the current population

Particle Swarm Algorithms (PSO) which model cooperative behavior of a swarm; e.g. a flock of birds

Ant Colony Algorithms (ACO) model the foraging behavior of ants

Genetic Algorithm

- The genetic algorithm (GA) is a method for solving an optimization problem starting from a set of completely random candidate solutions and searching for a solution using three principles of biological evolution:
  - Tournament selection

1	**2**	**3**	**4**	**3**	**4**	**5**	**6**

0	**9**	**6**	**7**	**2**	**8**	**1**	**6**

  - Binary crossover

0	**9**	**6**	**4**	**3**	**4**	**1**	**6**

  - Uniform mutation

1	**2**	**3**	**9**	**3**	**4**	**5**	**6**

  - Elitism (retain at least the best *n* unmodified individuals from the previous generation)
- There are both integer and real forms of the GA

Particle Swarm Optimization

The *Particle Swarm Optimization* (PSO) technique mimics the unpredictable motion of bird flocks while searching for food.

The initial population is randomly generated.

At a given iteration each particle is associated with a position vector and a velocity vector.

The formula for velocity update includes three terms with stochastic weights:

- The *inertial* component is proportional to the particle velocity in the preceding iteration
- The *cognitive* component is directed toward the best position experienced by the particle
- The *social* component is directed toward the best position yet located by any particle in the swarm.

At the end of the process the best particle is expected to contain the globally optimal values of the unknown parameters in the search space.

Particle Swarm Optimization

At the jth iteration, for particles i = 1, …, N

i) For i = 1, …, N: evaluate the objective function associated with particle i, $J^{(j)}(i)$

ii) Determine the best position ever visited (i.e. at any generation) by particle i, $y^{(j)}(i)$

iii) Determine the best position ever visited by any particle in the swarm, $Y^{(j)}(i)$

iv) Update the velocity vector for each particle:

$$w_k^{(j+1)}(i) = c_I w_k^{(j)}(i) + c_c\left[y_k^{(j)}(i) - x_k^{(j)}(i)\right] + c_s\left[Y_k^{(j)}(i) - x_k^{(j)}(i)\right], \quad k = 1,\dots,n$$

where the inertial, cognitive, and social weights have the following form:

$$c_I = \frac{1 + r_1(0,1)}{2}, \quad c_c = 1.49445 r_2(0,1), \quad c_s = 1.49445 r_3(0,1)$$

v) Update the position vector for each particle*: $x_k^{(j+1)}(i) = x_k^{(j)}(i) + w_k^{(j+1)}(i), \quad k = 1,\dots, n$

vi) Terminate when max number of iterations N_{IT} is reached.

Particle Swarm Optimization   Incorporating Constraints

- Two ways of dealing with problems with constraints: $g_j(\overline{x}) \le 0, \quad h_j(x) = 0$

1) Penalty method $fitness(\vec{x}) = \begin{cases} f(\vec{x}), & if\ \vec{x} \in F \\ f(\vec{x}) + penalty(\vec{x}), & otherwise \end{cases}$

$$penalty(\vec{x}) = \sum_{j=1}^{m} w_j \cdot \nu_j(\vec{x}) \qquad \nu_j(\vec{x}) = \begin{matrix} \max(0, g_j(\vec{x}_i)),\ if\ 1 \le j \le q \\ \left|h_j(\vec{x})\right|,\ if\ q+1 \le j \le m \end{matrix}$$

2) Multi-objective GA

$$fitness_1(\vec{x}) = f(\vec{x})$$

$$fitness_{i+1}(\vec{x}) = \nu_i(\vec{x})$$ for i = 1 to m

- Result is Pareto optimal set of solutions
  - Which solution is used for further analysis?

Test Case 1 Solved via Conventional Methods

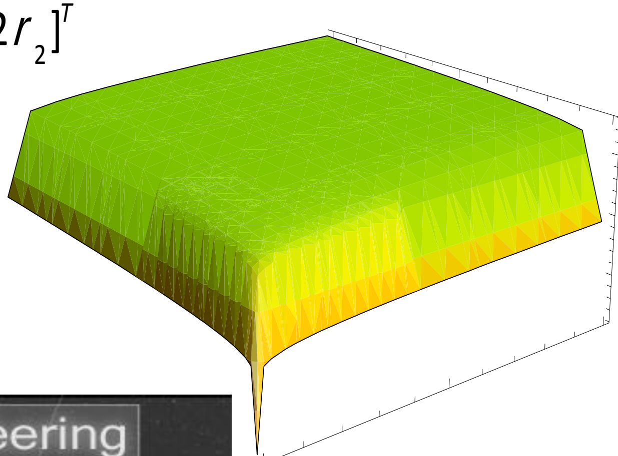
$$\min_{\rho_1,\rho_2} F(\rho) = \rho_1^2 + \rho_2^2 + \log(\rho_1\rho_2)$$

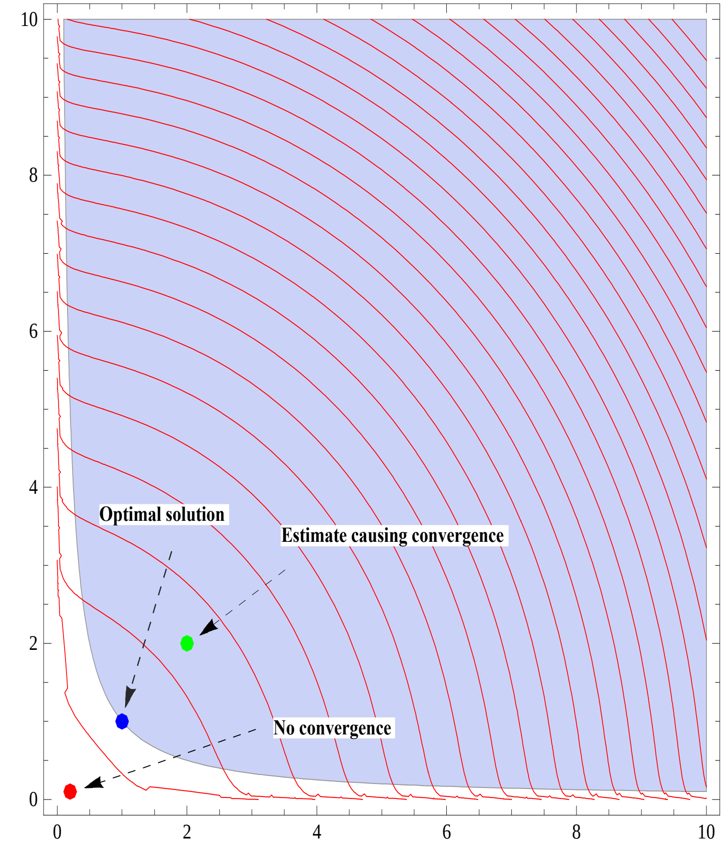
$$c(\rho) = 1 - \rho_1\rho_2 \le 0$$

$$0 \le \rho_1 \le 10,\ 0 \le \rho_2 \le 10$$

```
SNOPTA EXIT  20 -- the problem appears to be unbounded
SNOPTA INFO  21 -- unbounded objective
Problem name
No. of iterations                  4   Objective value      2.1460463782E+00
No. of major iterations            2   Linear objective     0.0000000000E+00
Penalty parameter          1.871E+01   Nonlinear objective  2.1460463782E+00
No. of calls to funobj            16   No. of calls to funcon             16
```

$$\nabla F(r) = \left[\frac{1}{r_1} + 2r_1, \frac{1}{r_2} + 2r_2\right]^T$$



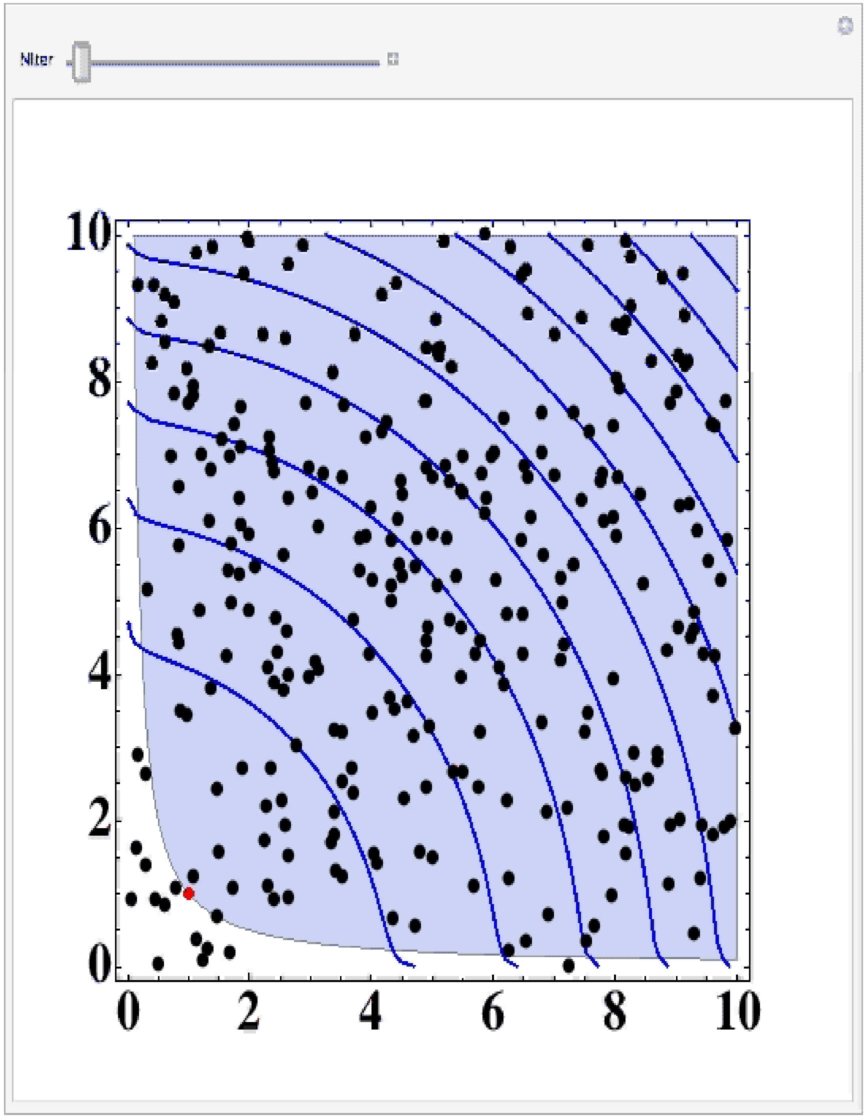


Static Optimization Test Case 1

$$\min_{\rho_1,\rho_2} F(\rho) = \rho_1^2 + \rho_2^2 + \log(\rho_1\rho_2)$$

$$c(\rho) = 1 - \rho_1\rho_2 \leq 0$$

$$0 \leq \rho_1 \leq 10,\ 0 \leq \rho_2 \leq 10$$

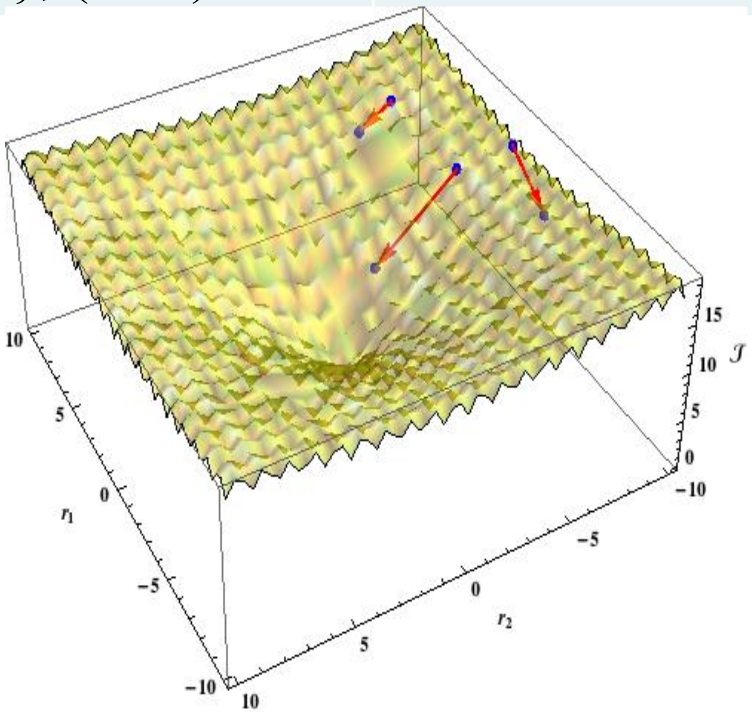


Test Case 2 Solved via Conventional Methods

Init Guess (Objective Val)	Converges To (Objective Val)	Major iterations
{2.24, -7.24}, (14.8)	{1.98, -2.97}, (7.96)	12
{6.59, -6.42}, (16.9)	{6.99, -4.99}, (14.1)	12
{1.24, -9.47}, (16.9)	{-2.00, -8.99}, (14.6)	11

Minimizing the Ackley Function Using SQP:
Global Minimum at {0,0}

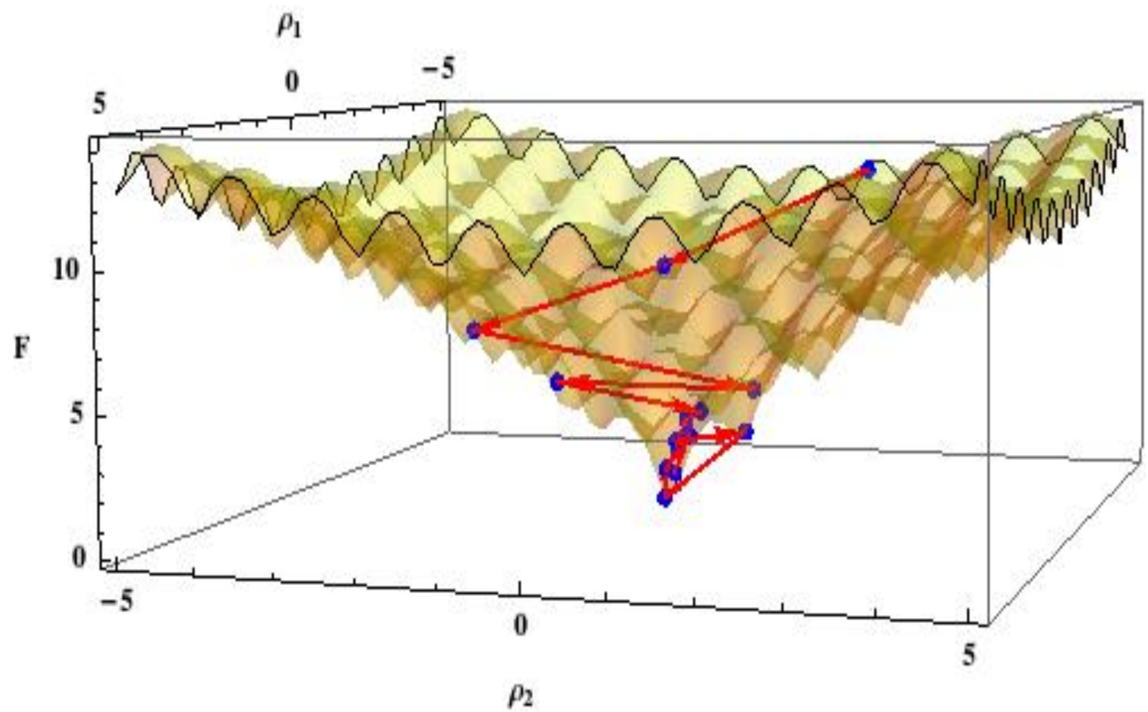
$$F(r) = -20\exp\left(-0.2\sqrt{0.5(r_1^2 + r_2^2)}\right)$$
$$-\exp(0.5(\cos(2\pi r_1)+\cos(2\pi r_2)))+20+e$$
$$F(0)=0$$



Test Case 2 Solved via PSO

$$\min_{\rho \in \Omega} F(\rho) = -20 \exp\left(-0.2\sqrt{0.5(\rho_1^2+\rho_2^2)}\right) - \exp\left(0.5(\cos(2\pi\rho_1)+\cos(2\pi\rho_2))\right)+20+e$$

$$\Omega = [-32.768, 32.768] \times [-32.768, 32.768]$$



- Advantages
  - Straightforward (possibly most-straightforward) to code
  - Don't need to know optimal control theory; don't need possibly difficult analytical differentiation
  - Requires no initial guess; the initial population is chosen randomly
  - More likely than other methods to locate the global minimum
- Disadvantages
  - The problem needs to be parameterized by a (relatively) small number of variables.
  - The methods depend on a number of user-selectable parameters and it is not *a priori* clear how these are chosen for a successful or efficient solution.
  - Likely to need explicit numerical integration of the EOM, which can be time-consuming
  - The solution will not be as accurate as that of the COV necessary conditions or a DT solution
  - Constraints need to be included via a penalty function method and this is especially problematic for equality constraints

Example: Max-Radius Orbit Transfer Using Solar Sail Solved Using PSO

Pradipto Ghosh, Ph. D. candidate, Univ. of Illinois 2013

Objective is to determine the solar sail orientation history (the control) in order to transfer the vehicle from a specified initial circular orbit to the largest possible coplanar circular orbit in a fixed time $t_f = 450$ days

$$\min_{a(\cdot)} J\left[x(\cdot),\ a(\cdot),\ t_f\right] = -r(t_f)$$

Subject to:

$$\dot{r} = v_r, \qquad r(0) = 1$$

$$\dot{q} = \frac{v_q}{r}$$

$$\dot{v}_r = \frac{v_q^2}{r} - \frac{m}{r^2} + a\frac{\cos^3 a}{r^2}$$

$$\dot{v}_q = -\frac{v_r v_q}{r} + a\frac{\sin a \cos^2 a}{r^2}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} v_q(t_f) - \sqrt{m/r(t_f)} \\ v_r(t_f) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

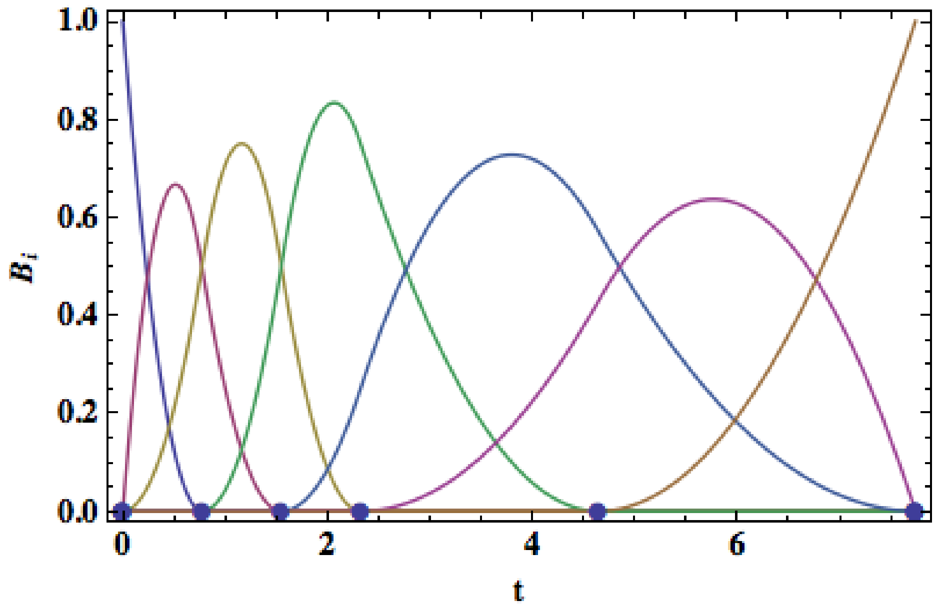
Example: Max-Radius Orbit Transfer Using Solar Sail Solved Using PSO (2)

The solution for α(t) was approximated as a sum of 7 quadratic splines.
The order of the spline $p_{s,k}$ is determined according to the "degree-parameter" $m_{s,k}$, which decides the B-spline degree of the **s**th control in the **k**th phase in the following fashion:

$$p_{s,k} = \begin{cases} 1 & \text{if} \quad -2 \le m_{s,k} < -1 \\ 2 & \text{if} \quad -1 \le m_{s,k} < 0 \\ 3 & \text{if} \quad 0 \le m_{s,k} < 1 \end{cases}$$

The optimal values for the 7 coefficients are found using PSO.

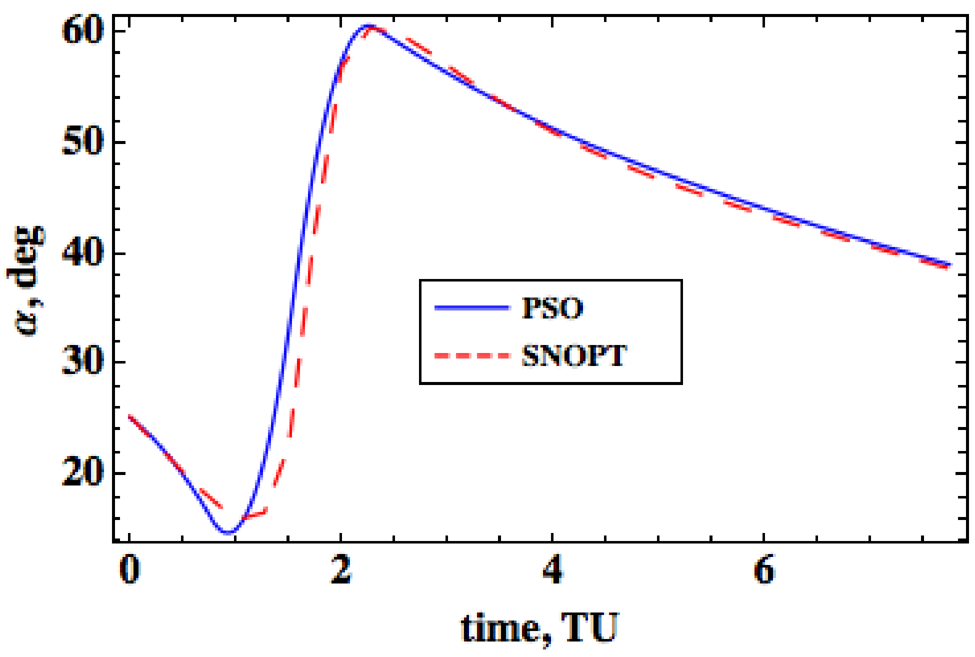
Parameter	Value
α_1	0.4377
α_2	0.3879
α_3	0.1665
α_4	1.096
α_5	0.9228
α_6	0.7494
α_7	0.6798
$m_{1,1}$	-0.3364

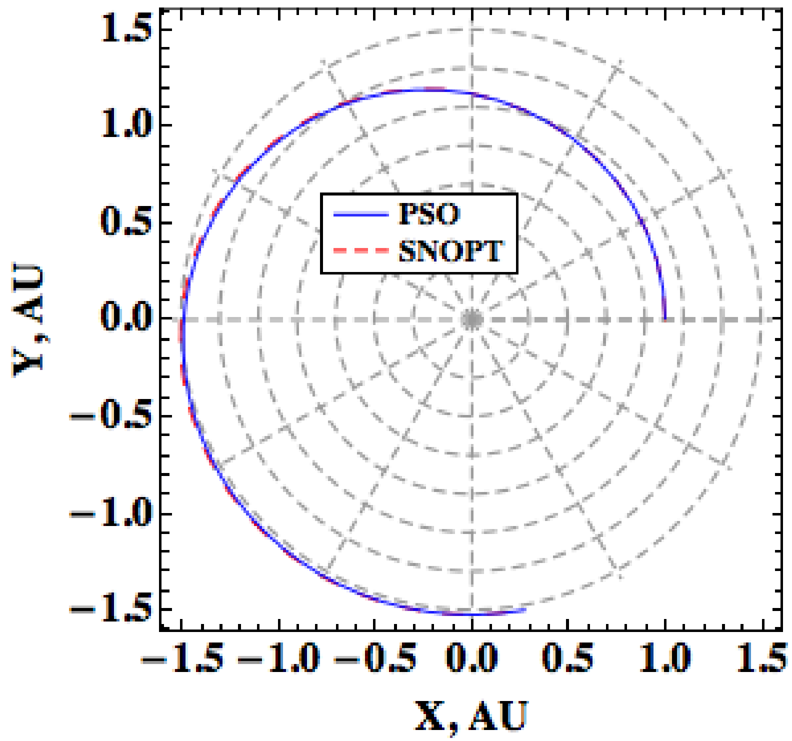


Example: Max-Radius Orbit Transfer Using Solar Sail Solved Using PSO (3)

The PSO solution is then used as the initial guess for a more-accurate solution using a direct approach (GPM) with NLP solver SNOPT.

The results are compared in these figures.





So What is the Best Solution Method?

The answer is definitely problem-dependent, with the most important consideration being (in my judgment) whether the trajectory uses impulsive thrust or low-thrust (electric) propulsion.

The impulsive case, even with planetary flybys, is a parameter optimization problem and can be solved very efficiently using metaheuristics, esp. PSO or GA + PSO.

The low-thrust case is a continuous optimization problem. It must somehow be converted into a parameter optimization problem. The resulting problem is usually orders of magnitude larger than that of the impulsive case.

My work (with many students) in recent years suggests that for trajectories that include low-thrust arcs *the best approach is an Metaheuristic Algorithm alone or in combination with a Direct Transcription method.*

An example is the immediately preceding solar sail trajectory problem. Note that the PSO solution is very good and the direct solver converges very quickly using the PSO solution as an initial guess!

Why?

- Both Hamiltonian (COV-based) methods and DT methods require an "initial guess". This is often problematic.

  - Approximate solution obtained with EA's or heuristics can become the initial guess for a more-accurate COV-based or DT solution.

- Both Hamiltonian methods and DT methods are local.

  - Heuristic methods are much more likely to find a global minimum.

- For the special case of mission planning problems, a successful solution strategy has been a outer-loop solver/inner-loop solver.

  - The outer-loop solver chooses the discrete decision parameters and is well suited to a GA.

- As a "bonus", heuristic solutions are much easier to program, e.g. they need no gradient or Jacobian information.

Lecture 2: Methods for Optimizing Interplanetary Trajectories II / Application to the Problem of NEA Deflection

Metaheuristic methods for solution of optimal spacecraft trajectory problems: as initial guess for a more precise method *or* as solution in own right

Need different approaches for “naturally discrete” problems/ continuous-thrust problems

Examples

Proposed methods of asteroid deflection: collision, nuclear explosion, mass drivers, gravity tractor. Advantages and disadvantages.

The objective: maximizing deflection (easy) vs. maximizing displacement from Earth surface (complicated)

Using the state transition matrix to optimize deflection via collision

Conversion of the Optimal Control Problem for Solution via EA

Using a EA requires that the problem be formulated as a “few parameter” problem in contrast to direct transcription formulations in which there are 100’s to several 1000’s of decision parameters.

All space trajectory problems are continuous, but the associated optimal control problem may be discrete or continuous.

“Naturally Discrete” Cases	**Continuous Þ Discrete Cases**
Hohmann transfer	Low-thrust transfers
Lambert problem transfer	Impulsive + low-thrust transfers
Multiple Gravity Assist (MGA)	Lyapunov periodic orbits

Example: Hybrid Optimal Control Using Nested Loops – Minimum-Fuel Interplanetary Trajectory with Flybys and Deep Space Maneuvers

Jacob Englander, Ph. D. thesis 2013
The problem is formulated as a *hybrid optimal control problem* (HOCP).

No *a priori* information about the solution is provided, only a range of dates is given for departure from Earth and arrival at the target planet.

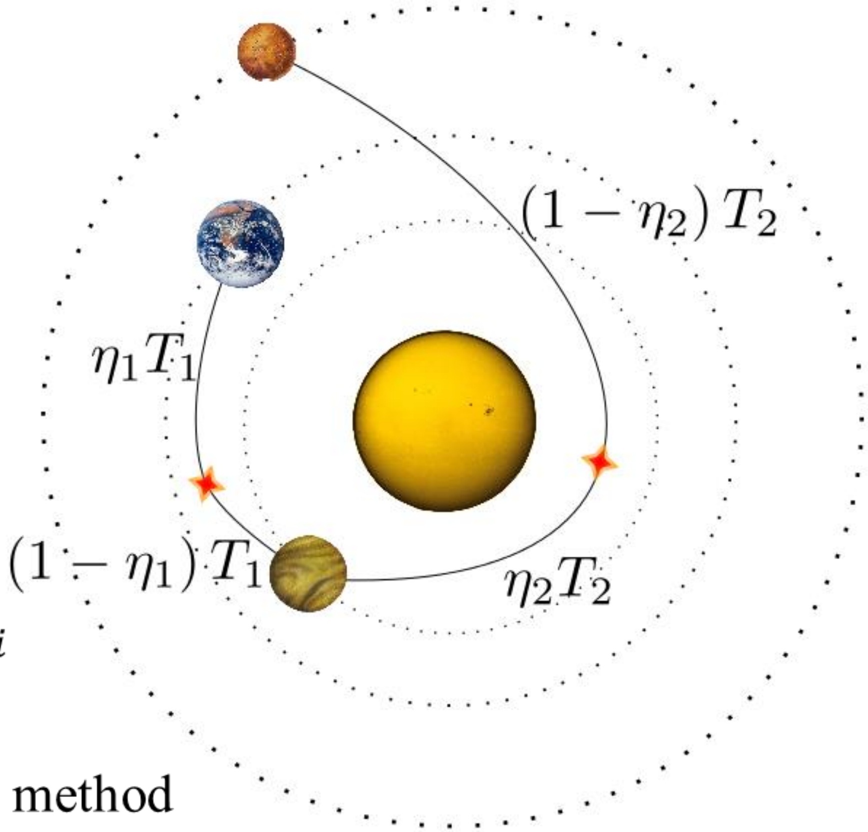
An outer-loop solver determines the optimal number and sequence of flybys. Each planet in the solar system is given a number (Mercury = 1 … Neptune = 8; 0 and 9-15 are null codes). A binary GA determines the prospective sequences; e.g. [2 3 11 15 5 9 12] would be an Earth departure followed by flybys at Venus (2), Earth (3) and Jupiter (5) then arrival at the target planet. The null codes allow as many as 7 flybys in the mission.

For each sequence an inner-loop solver, using differential evolution (DE), determines the optimal parameters: dates of all important events, flyby periapse radii; locations and directions of deep space maneuvers.

The optimal cost from the inner-loop solution is returned to the outer-loop GA. The GA then quickly identifies poorly-performing sequences.

Example: Hybrid Optimal Control Using Nested Loops – Minimum-Fuel Interplanetary Trajectory with Flybys and Deep Space Maneuvers (2)

- Allow one deep-space maneuver in each phase, no maneuvers at flyby periapse
- Optimizer Chooses:
  - Launch date
  - Initial hyperbolic $\boldsymbol{v}_\infty$
  - For each phase:
    - Flight time T_i
    - Burn index η_i
    - B-plane insertion angle γ_i
    - Flyby periapse distance ratio R_{pi}



- In each phase:
  - Propagate to burn point via Kepler's method
  - Find trajectory from burn point to end point via Lambert's method

Example: Hybrid Optimal Control Using Nested Loops – Minimum-Fuel Interplanetary Trajectory with Flybys and Deep Space Maneuvers (3)

Option	Value
Arrival type	insertion into orbit about Saturn
Semi-major axis of capture orbit about Saturn	5447500 km
Eccentricity of capture orbit about Saturn	0.998
Launch window open date	4/7/1997
Launch window close date	1/1/2000
Flight time upper bound	10 years
Maximum number of flybys	8

Example: Hybrid Optimal Control Using Nested Loops – Minimum-Fuel Interplanetary Trajectory with Flybys and Deep Space Maneuvers (4)

Automated Choice of Parameter Bounds for DE Solver

Parameter	**Upper Bound**	**Lower Bounds**
Launch Date	User defined	User defined
Stay time between journeys	User defined	User defined
Right Ascension of Launch Asymptote	0.0	2π
Declination of Launch Asymptote	User defined	User defined
v_∞ at launch	0.0	User defined
For each phase:		
Flight Time		
repeated flyby of same planet	$T/2$	$5T$
outermost body has $a < 2$ *AU*	0.1 $min\ (T_1, T_2)$	1.5 $max\ (T_1, T_2)$
outermost body has $a \geq 2$ *AU*		$max\ (T_1, T_2)$
	Maximum of 600 days	Minimum of 1000 days
Burn index η	0.1	0.9
B-plane insertion angle γ	$-\pi$	π
Flyby periapse distance ratio R_p		
rocky planets	1.05 times radius of planet	10 times radius of planet
gas giants	1.05 times radius of planet	300 times radius of planet

Example: Hybrid Optimal Control Using Nested Loops – Minimum-Fuel Interplanetary Trajectory with Flybys and Deep Space Maneuvers (5)

The 20 best sequences found by the outer-loop GA (in 46 hours on Intel Core i7)

Sequence	Cost (km/s)
EVVEJS	1.01
EVVES	2.31
EEVVES	2.49
EMEJS	2.64
EEMES	3.09
EVMES	3.13
EVEES	3.19
EVES	3.19
EMES	3.23
EMVVES	3.30
EMVJS	3.39
EVEJS	3.39
EVVJS	3.49
EVMVES	3.65
EEVMES	3.75
EVMEES	3.83
EEVVEJS	3.99
EVVVES	4.04
EEVES	4.09
EVVVEJS	4.10

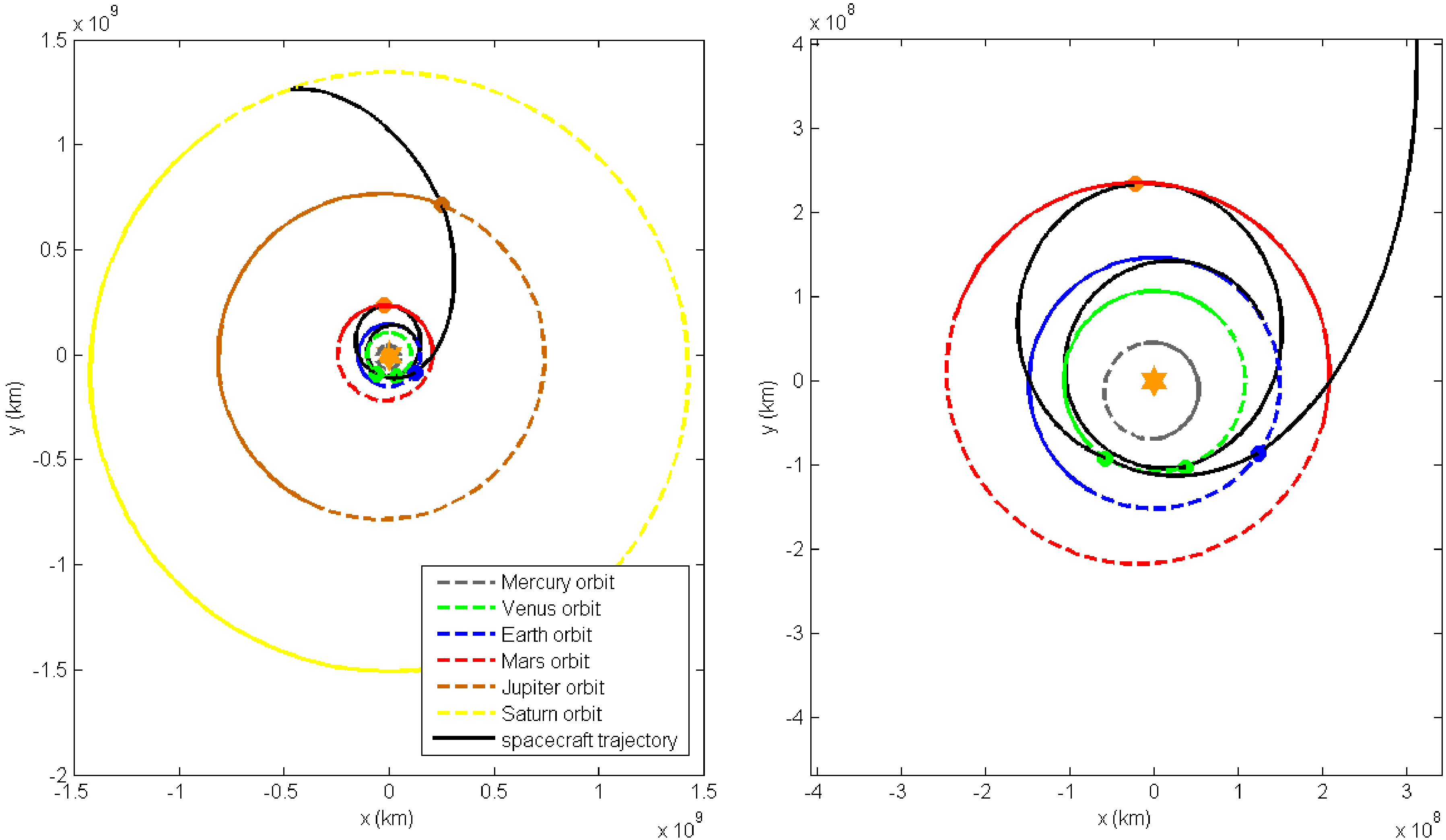
Example: Hybrid Optimal Control Using Nested Loops – Minimum-Fuel Interplanetary Trajectory with Flybys and Deep Space Maneuvers (6)

Itinerary found by the optimizer for the Cassini MGA-DSM mission.

Date	Event	Location	ΔV (km/s)	Flyby altitude (km)	γ (degrees)
10/22/1997	launch	Earth	4.25081	-	-
5/3/1998	flyby	Venus	-	3952	-97.7
12/4/1998	burn	deep-space	0.389606	-	-
6/23/1999	flyby	Venus	-	728	-112.8
8/17/1999	flyby	Earth	-	924	-88.9
1/15/2001	flyby	Jupiter	-	9.53E+06	-88.8
10/22/2004	insertion	Saturn	0.619974	-	-

Event	Optimal MGA-1DSM Solution	Actual Cassini Mission
Launch	10/22/1997	10/15/1997
Venus flyby 1	5/3/1998	4/26/1998
Venus flyby 2	6/23/1999	6/24/1999
Earth flyby	8/17/1999	8/18/1999
Jupiter flyby	1/15/2000	12/30/2000
Saturn orbit insertion	10/22/2004	7/1/2004
Cost	1010 m/s	1079 m/s

Example: Hybrid Optimal Control Using Nested Loops – Minimum-Fuel Interplanetary Trajectory with Flybys and Deep Space Maneuvers (7)



Conversion of the Continuous Optimal Control Problem for Solution via EA

Problems that are not "naturally discrete", e.g. problems using low-thrust propulsion, must be recast as depending on a small number of decision parameters. There are a number of ways to accomplish this:

i) The thrust magnitude and thrust pointing time histories can be described by polynomials, splines, or Fourier series. The decision parameters are then the coefficients. Optimal solar sail control example.

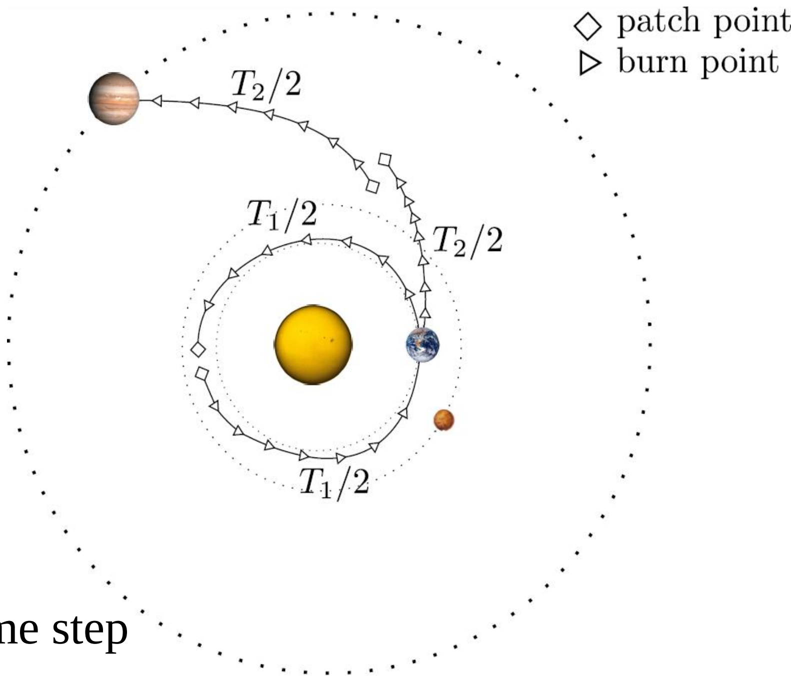
ii) The arcs during which thrust is applied can be modeled using "shape-based" methods in which a GA chooses the optimal parameters describing the shape. Optimal NEA deflection via impactor example.

iii) The Sims-Flanagan approximation can be used. In this approximation a continuous thrust arc is modeled as a sequence of discrete, small $\Delta V's$ Bepi Columbo-like mission

iv) The problem can be formulated using a Hamiltonian method so that the unknowns become unknown initial values of the Lagrange multipliers (costates) of the problem. The states are then found by numerical integration and the control is found using Pontryagin's principle. Low-thrust circle-circle rendezvous example.

Multiple Gravity Assist with Low Thrust (MGA-LT)

- Break mission into phases. Each phase starts and ends at a body.
- Sims-Flanagan Transcription
  - Break phases into time steps
  - Insert a small impulse in the center of each time step, with bounded magnitude
  - Optimizer Chooses:
    - Launch date
      - For each phase:
        - » Initial velocity vector
        - » Flight time
        - » Thrust-impulse vector at each time step
        - » Mass at the end of the phase
        - » Terminal velocity vector



- Propagate forward and backward from phase endpoints to a “match point”
- Enforce nonlinear state continuity constraints at match point
- Enforce nonlinear velocity magnitude and altitude constraints at flyby

Example of MGA-LT: "BepiColombo"- like Mission

- Objective is to travel from Earth to Mercury (within a specified range of dates) and maximize payload delivered.

- "Outer-loop" GA chooses number and sequence of planetary flybys

- "Inner-loop" trajectory optimizer first uses monotonic basin hopping (MBH) a heuristic method, to find an approximate solution to be used as an initial guess.

- The problem parameters are the launch date, the time of flight, the magnitude and direction of the departure impulse at Earth, and then all of the Sims-Flanagan parameters that describe each thrust arc (shown in the previous slide).
  There are 191 decision variables and 95 constraints

- Then a direct solver using NLP (SNOPT) finds an accurate solution using the MBH solution as its initial guess.

Example of MGA-LT: "BepiColombo"- like Mission Problem Assumptions

Option	Value
Arrival type	intercept (match position) with bounded v_∞
Maximum arrival v_∞	0.5 km/s
Launch window open date	8/1/2009
Launch window close date	4/27/2012
Flight time upper bound	15 years
Propulsion type	Fixed Isp and thrust
Thrust (N)	0.34
Isp (s)	3200
Initial mass (kg)	1300
Maximum Δv_{LV} (km/s)	1.925
Number of time steps per phase	10
Maximum number of Flybys	8
GA Population Size	100
Inner-Loop run time per sequence	2 hours

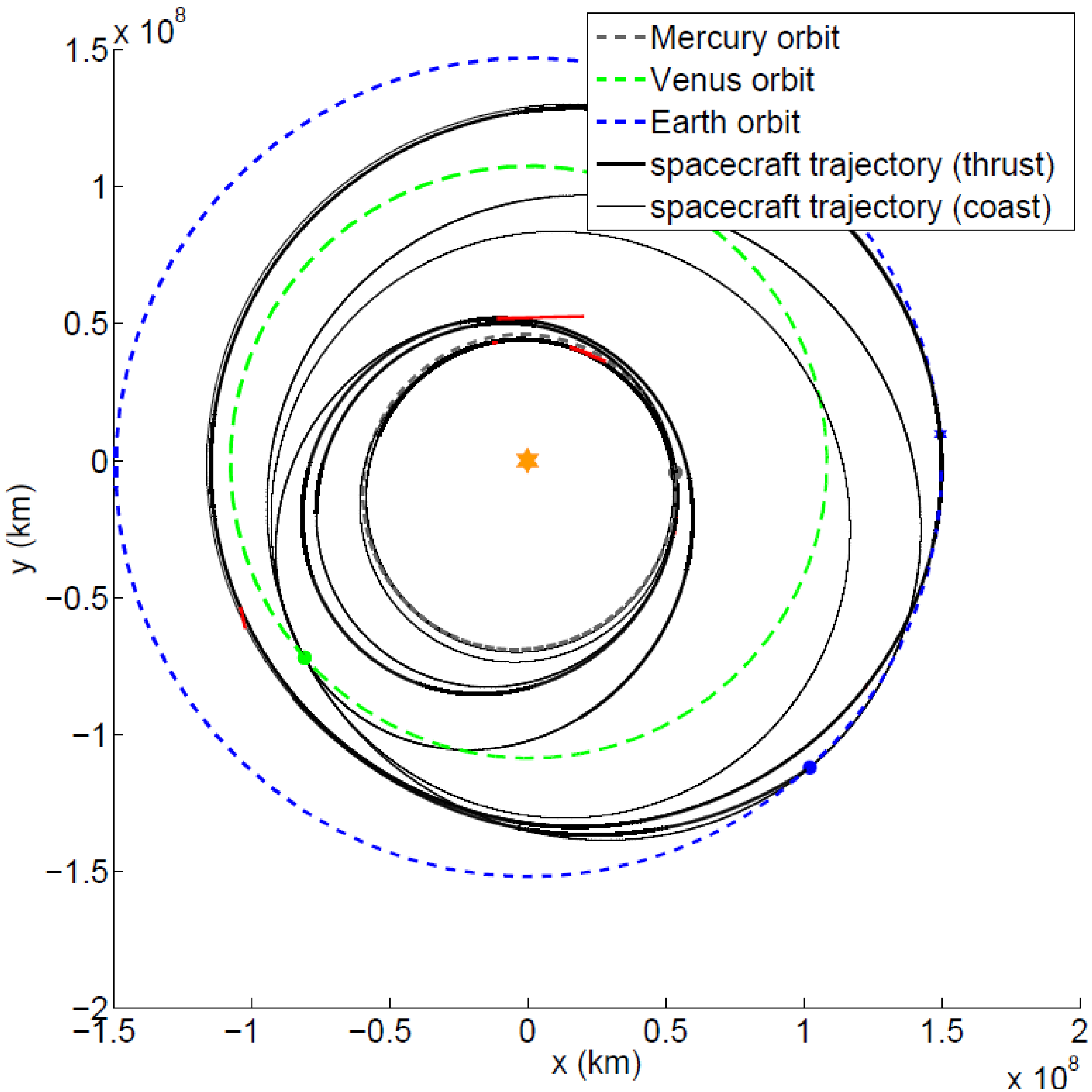
Objective: maximize mass delivered to Mercury

No other information is supplied by the user

Example of MGA-LT: "BepiColombo"-like Mission
Best 20 Solutions Found by the GA

Sequence	Final Mass (kg)
EEVVYY	1112
EVVYY	1077
EEEVVYY	1077
EEVVVYY	1076
EEVYYY	1061
EEVYY	1045
EMVVYY	1038
EEVVY	1030
EEEEVYY	1030
EEEVYY	1026
EEVY	1024
EVVY	1020
EEVEVYY	1013
EVVVY	1006
EVYY	998
EMEVVYY	972
EVYYY	970
EVY	964
EEYYY	937
EMEVY	930

Example of MGA-LT: "BepiColombo"-like Mission Trajectory



Example of MGA-LT: “BepiColombo”-like Mission Itinerary

Date	Event	Location	Mass (kg)	Flyby altitude (km)
9/26/2011	launch	Earth	1300	-
8/3/2016	flyby	Earth	1272	21945
10/7/2017	flyby	Venus	1272	3895
2/19/2019	flyby	Venus	1272	303
4/10/2021	flyby	Mercury	1159	122
5/23/2022	arrival	Mercury	1112	-

Yam *et al* found a best cost of 1064 kg for an EVVYYY sequence

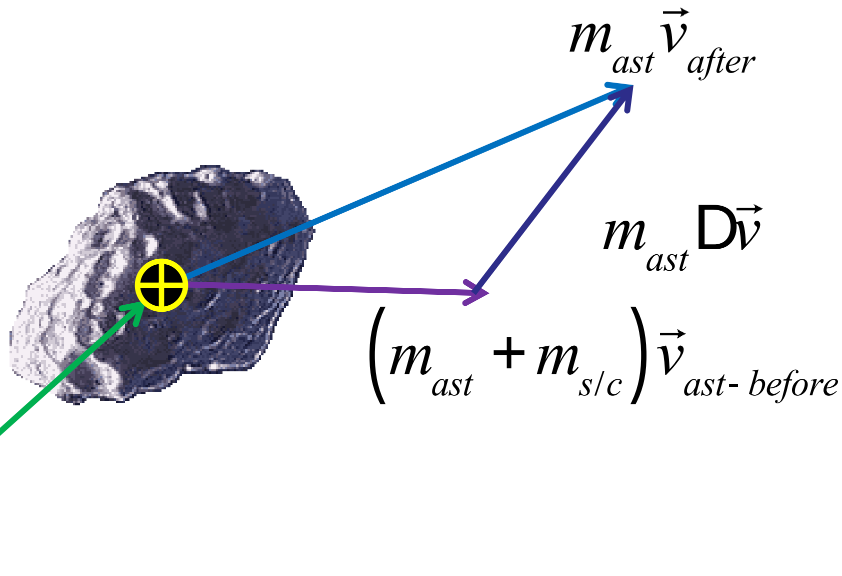
Methods Suggested for Asteroid Deflection

Method	**Impact**	**Nuclear Explosion**	**Gravity Tractor**	**Yarkovsky Effect**	**Surface Mass Driver**
Interception	X				
Rendezvous		X	X	X	X
Impulsive or Low-Thrust	Either or Both	Either or Both	Either or Both	Either or Both	Either or Both
Planetary Flybys	Possibly	Possibly	Possibly	Possibly	Possibly
Effect	Sudden	Sudden	Slow	Slow	Slow

Advantages & Disadvantages

Method	**Impact**	**Nuclear Explosion**	**Gravity Tractor**	**Yarkovsky Effect**	**Surface Mass Driver**
Comparatively simple	X			X	
Precisely controllable			X		X
Requires large masses			X		X
Result requires asteroid characterization	Possibly	X			Possibly
Mass used efficiently	X	X		X	

Kinetic Impactor Dynamics



Conservation of momentum:

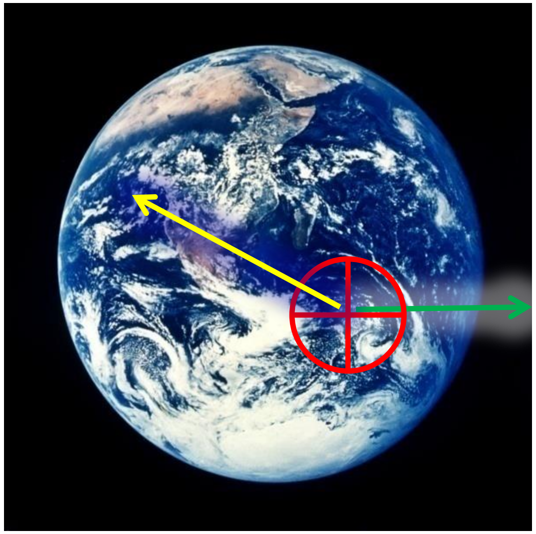
$$m_{ast}\vec{v}_{ast-before} + m_{s/c}\vec{v}_{s/c} = (m_{ast} + m_{s/c})\vec{v}_{after}$$

Note that since $m_{s/c}$ is very small relative to m_{ast}, the applied Δv will be small, on the order of mm/s

Problem Definition

- Objective: Using a kinetic impactor, maximize the distance by which the asteroid misses the earth
- This is not necessarily the same thing as maximizing the deflection distance or the asteroid's orbital energy!

Direction of largest possible deflection



Direction of largest miss distance

Kinetic Impactor Dynamics – The State Transition Matrix

$$\begin{bmatrix} \delta\vec{r} \\ \delta\vec{v} \end{bmatrix} = \Phi\left(t,t_0\right)\begin{bmatrix} \delta\vec{r}_0 \\ \delta\vec{v}_0 \end{bmatrix} = \begin{bmatrix} \tilde{R} & R \\ \tilde{V} & V \end{bmatrix}\begin{bmatrix} \delta\vec{r}_0 \\ \delta\vec{v}_0 \end{bmatrix}$$

$$\delta\vec{r}\left(t\right) = \left[R\right]\delta\vec{v}_0\left(t_0\right)$$

$$\left[R\right] = \frac{r_0}{\mu}\left(1 - F\right)\left[\left(\vec{r} - \vec{r}_0\right)\vec{v}_0^{\,T} - \left(\vec{v} - \vec{v}_0\right)\vec{r}_0^{\,T}\right] - \frac{C}{\mu}\vec{v}\vec{v}_0^{\,T} + G\left[I_3\right]$$

$$F = 1 - \frac{r}{p}\left(1 - \cos\theta\right)$$

$$G = \frac{1}{\sqrt{\mu}}\left[\frac{r r_0}{\sqrt{p}}\sin\theta\right]$$

$$C = \frac{1}{\sqrt{\mu}}\left[3U_5 - \chi U_4 - \sqrt{\mu}\left(t - t_0\right)U_2\right]$$

$$U_k = U_k\left(\chi,\alpha\right)$$

$$\chi = \sqrt{a}\left(E - E_0\right)$$

$$\alpha = {}^{1}\!/_{a}$$

The unperturbed asteroid's position on the date of close approach is known.

The asteroid is perturbed by the kinetic impactor, and the state transition matrix is solved analytically to find the difference between the asteroid and its unperturbed reference position on the date of close approach.

The longer the asteroid coasts after receiving impulse from the interceptor, the more it will deviate from its reference course. Thus, in general, the earlier the interceptor hits the asteroid, the farther away the asteroid will pass the Earth.

Maximization of the Deflection via Nuclear Impulse (1)

The state transition matrix $F(t, t_0)$ determines the perturbation in position and velocity:

$$\begin{bmatrix} d\bar{r} \\ d\bar{v} \end{bmatrix} = F(t, t_0) \begin{bmatrix} d\bar{r}_0 \\ d\bar{v}_0 \end{bmatrix} = \begin{bmatrix} \tilde{R} & R \\ \tilde{V} & V \end{bmatrix} \begin{bmatrix} d\bar{r}_0 \\ d\bar{v}_0 \end{bmatrix}$$

Therefore:

$$d\bar{r}(t) = R\, d\bar{v}_0(t_0)$$

where t is the time of close approach and t_0 is the time of interception, where

$$[R] = \frac{r_0}{m}(1 - F)\left[(\bar{r} - \bar{r}_0)\bar{v}_0^T - (\bar{v} - \bar{v}_0)\bar{r}_0^T\right] + \frac{C}{m}\bar{v}\,\bar{v}_0^T + G\,[I]$$

$$F = 1 - \frac{r}{p}(1 - \cos q)\,, \quad \cos q = \frac{\bar{r} \times \bar{r}_0}{r\, r_0}$$

$$G = \frac{1}{\sqrt{m}}\left[\frac{r\, r_0}{\sqrt{p}} \sin q\right]$$

Maximization of the Deflection via Nuclear Impulse (2)

and

$$C = \frac{1}{\sqrt{m}}\left[3\,U_5 - c\,U_4 - \sqrt{m}\,(t - t_0)\,U_2\right]$$

where

$$c = \sqrt{a}\,(E - E_0)$$

and

$$U_1(c, a),\ U_2(c, a),\ U_3(c, a),\ U_4(c, a),\ U_5(c, a)$$

are the "Universal Functions" (cf. Battin's book), where $a = 1/a$

Maximization of the Deflection via Nuclear Impulse (3)

Want maximum deflection at close approach time t, i.e. max $\left|d\bar{r}(t)\right| = \max\left(\left[R\right] d\bar{v}_0\right)$

This is equivalent to maximizing $d\bar{v}_0^T \left[R\right]^T \left[R\right] d\bar{v}_0$

This quadratic form is maximized, for given $\left|d\bar{v}_0\right|$, if $d\bar{v}_0$ is chosen parallel

to the eigenvector of $\left[R\right]^T \left[R\right]$ conjugate to the largest eigenvalue of $\left[R\right]^T \left[R\right]$.

This yields the optimal direction for the perturbing velocity impulse $d\bar{v}_0$, (in the space-fixed XYZ basis).

Can then express $d\bar{v}_0$ in asteroid-fixed radial, transverse, normal basis as

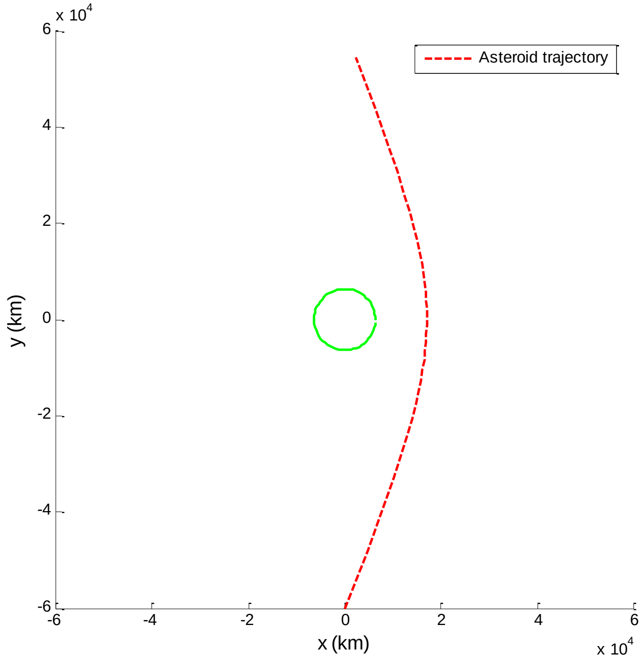
$$d\bar{v}_{0\,\mathrm{RTN}} = \begin{bmatrix} cqcW - cisWsq & cqsW + cicWsq & sisq \\ -sqcW - cisWcq & -sqsW + cicWcq & sicq \\ sisW & -sicW & ci \end{bmatrix} d\bar{v}_{0\,\mathrm{XYZ}}$$

Maximization of the Miss Distance from Earth Surface (1)

As shown in a previous slide, maximizing deflection magnitude is straightforward, using the system state transition matrix.

However the real objective is to maximize the miss distance altitude, which is the same as maximizing the miss distance radius (from the center of the Earth).

This is accomplished by maximizing the perigee radius of the asteroid's hyperbolic flyby.



Maximization of the Miss Distance from Earth Surface (2)

It is first necessary to determine the heliocentric position and velocity at entry onto the Earth arrival hyperbola:

$$\begin{bmatrix} \vec{r}\left(t_f\right) \\ \vec{v}\left(t_f\right) \end{bmatrix} = \Phi\left(t_f, t_{intercept}\right)\begin{bmatrix} 0 \\ \delta\vec{v}_0 \end{bmatrix} + \begin{bmatrix} \vec{r}_{reference}\left(t_f\right) \\ \vec{v}_{reference}\left(t_f\right) \end{bmatrix}$$

Then, the miss distance from the Earth's center may be found as:

$$r_{miss} = a_{f/b}\left(1 - e_{f/b}\right)$$

where

$$a_{f/b} = - m_{earth} \Big/ v_{\text{¥}}^2$$

v_∞ is the hyperbolic approach velocity given by

$$v_\infty = \left\| v_{deflected}\left(t_f\right) - v_{earth}\left(t_f\right) \right\|$$

and $e_{f/b}$ is the eccentricity of the hyperbolic trajectory given by

$$e_{f/b} = \csc\left(d \big/ 2 \right) \qquad \text{where} \qquad \delta = 2\tan^{-1}\left(\frac{\mu_{earth}}{\left\| \vec{r}_{deflected}\left(t_f\right) - \vec{r}_{earth}\left(t_f\right) \right\| \cdot v_\infty^2} \right)$$

Lecture 3: Asteroid Deflection/ Characterization by Sample Return

Optimizing asteroid deflection (magnitude) via nuclear explosion using a low-thrust spacecraft

Optimizing asteroid deflection (from Earth surface) via collision using a low-thrust spacecraft

Optimizing a manned reconnaissance/sample return mission to an asteroid

Examples and results

Example: Deflection of PHA Using Nuclear Impulse and a Low-Thrust Spacecraft

- Simulation assumes use of current technology
- Low-thrust propulsion is used because of its great efficiency
- It is assumed that the nuclear explosion can be exploded at the moment of close approach, i.e. rendezvous is not required
- It will be shown that use of low-thrust electric propulsion yields a dramatic increase in payload delivered to the asteroid
- The problem is formulated using the Gauss variational equations in (singularity-- free) equinoctial elements:

$$a,\ P_1 = e\sin\varpi,\ P_2 = e\cos\varpi,\ Q_1 = \tan\frac{i}{2}\sin\Omega,\ Q_2 = \tan\frac{i}{2}\cos\Omega,\ L$$

- The optimal control problem is constructed using a direct transcription method (collocation with 5th degree Gauss-Lobatto defects) and solved using a NLP problem solver (NPSOL).
- The control variables are the in-plane (azimuthal) thrust pointing angle and the out-of plane thrust pointing angle. The optimizer may also choose the direction of the hyperbolic departure from Earth.

“Independence” of Trajectory and Deflection

<u>In principle</u>

The solution to the “complete” problem involves finding optimal values (or time histories) of all free parameters, from Earth departure to interception, e.g. optimal departure point in Earth orbit, departure direction, optimal thrust pointing history, arrival date, direction of application of impulse, etc.

<u>But</u>

The locus of points where a deflection impulse can be applied is the asteroid orbit!

So best direction of application of impulse and resulting maximum deflection can be found independently of the interception trajectory!

<u>However</u>

It is still necessary to intercept the asteroid optimally – an optimal low-thrust trajectory is found to each candidate interception point.

Initial Condition Constraints

The initial condition constraints yield the following 6 scalar equations:

$$a_E \cos \theta_E - \frac{r\left[\cos L + (Q_2^2 - Q_1^2)\cos L + 2\,Q_1\,Q_2 \sin L\right]}{1 + Q_2^2 + Q_1^2} = 0$$

$$a_E \sin \theta_E - \frac{r\left[\sin L - (Q_2^2 - Q_1^2)\sin L + 2\,Q_1\,Q_2 \cos L\right]}{1 + Q_2^2 + Q_1^2} = 0$$

$$\frac{2r\left[\,Q_2 \sin L - Q_1 \cos L\right]}{1 + Q_2^2 + Q_1^2} = 0$$

$$\frac{\sqrt{\mu/p}\left[\sin L + (Q_2^2 - Q_1^2)\sin L - 2\,Q_1\,Q_2 \cos L + P_1 - 2P_2Q_1Q_2 + (Q_2^2 - Q_1^2)P_1\right]}{1 + Q_2^2 + Q_1^2}$$

$$-v_E \sin \theta_E + v_{\infty/E} \cos \gamma_0 \sin(\beta_0 - \theta_E) = 0$$

$$\frac{\sqrt{\mu/p}\left[-\cos L + (Q_2^2 - Q_1^2)\cos L + 2\,Q_1\,Q_2 \cos L - P_2 + 2P_1Q_1Q_2 + (Q_2^2 - Q_1^2)P_2\right]}{1 + Q_2^2 + Q_1^2}$$

$$+v_E \cos \theta_E + v_{\infty/E} \cos \gamma_0 \cos(\beta_0 - \theta_E) = 0$$

$$\frac{2\sqrt{\mu/p}\left[\,Q_2 \cos L + Q_1 \sin L - P_2 + P_1Q_1 + Q_2\,P_2\right]}{1 + Q_2^2 + Q_1^2} - v_{\infty/E} \sin \gamma_0 = 0$$

where all unsubscripted variables and elements refer to the orbit of the interceptor spacecraft, θ_E is the true longitude of the Earth, v_E is the circular velocity of the Earth and all quantities are evaluated at t = 0.

Terminal Constraints (Interception)

The terminal constraint yields the following 3 scalar equations:

$$r_A[\cos q_A \cos W_A - \cos i_A \sin W_A \sin q_A] - \frac{r\left[\cos L + (Q_2^2 - Q_1^2)\cos L + 2Q_1 Q_2 \sin L\right]}{1 + Q_2^2 + Q_1^2} = 0$$

$$r_A[\cos q_A \cos W_A - \cos i_A \sin W_A \sin q_A] - \frac{r\left[\sin L - (Q_2^2 - Q_1^2)\sin L + 2Q_1 Q_2 \cos L\right]}{1 + Q_2^2 + Q_1^2} = 0$$

$$r_A \sin i_A \sin q_A - \frac{2r\left[Q_2 \sin L - Q_1 \cos L\right]}{1 + Q_2^2 + Q_1^2} = 0 \qquad (12)$$

where all unsubscripted variables refer to the orbit of the interceptor spacecraft and all quantities are evaluated at t_{Final}.

Target is Asteroid 1991RB

Orbit of Asteroid 1991RB:

a = 1.4524 AU

e = .4846

i = 19.578 °

W= 359.738 °

w = 68.703 °

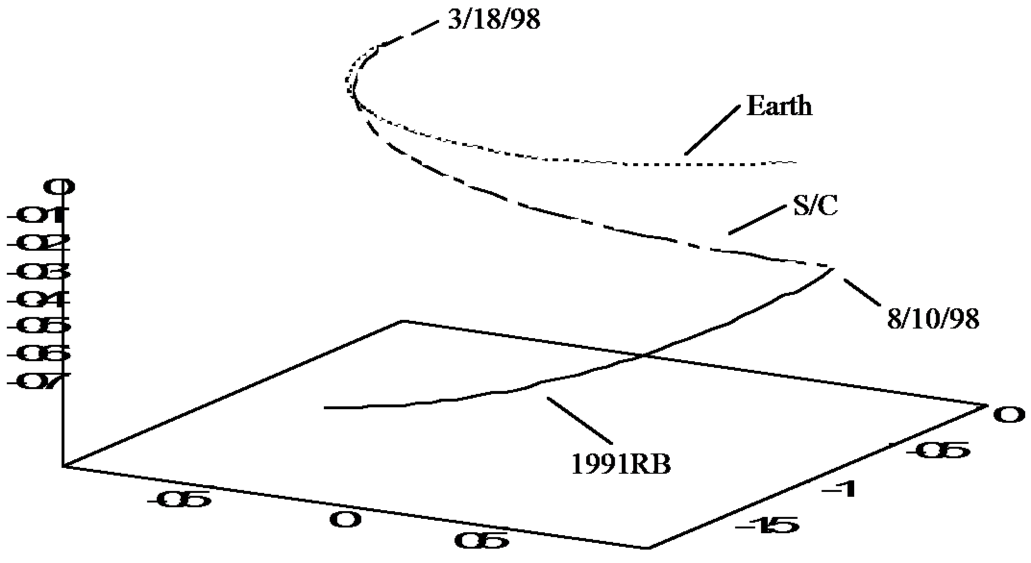
M = 225.871 °

At epoch 3/18/1998

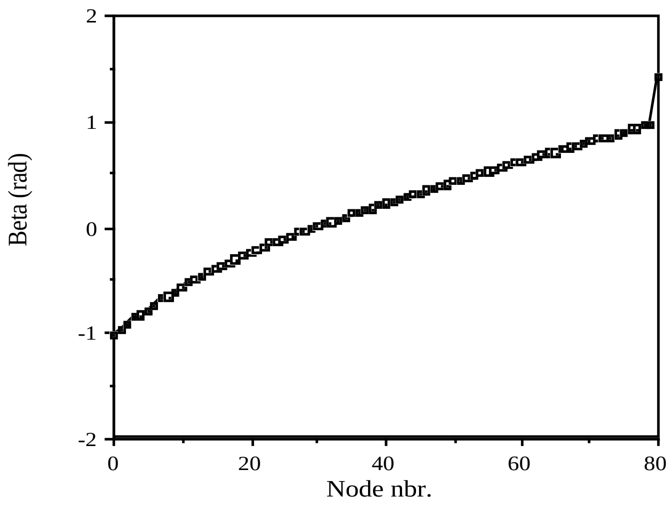
Asteroid 1991RB had a close approach to Earth of .0401 AU (= 15.62 Lunar Distances) on 9/18/1998.

A Sample Trajectory

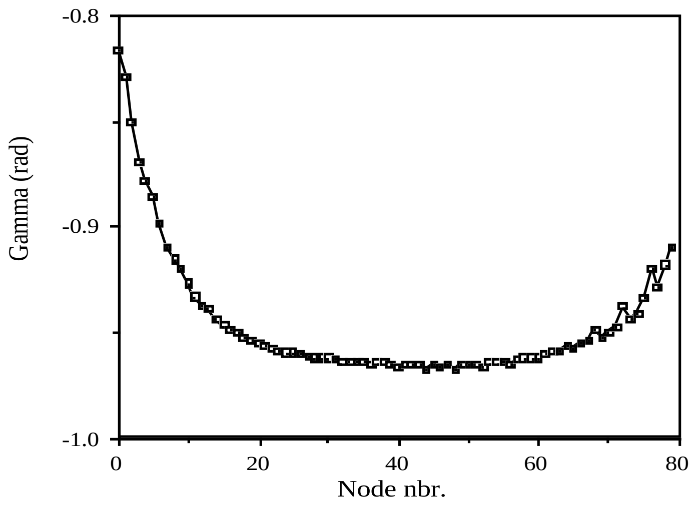
Example trajectory: launch 6 months before close approach; interception 38 days before close approach



History of the Thrust Pointing Angles for Interception of 1991RB

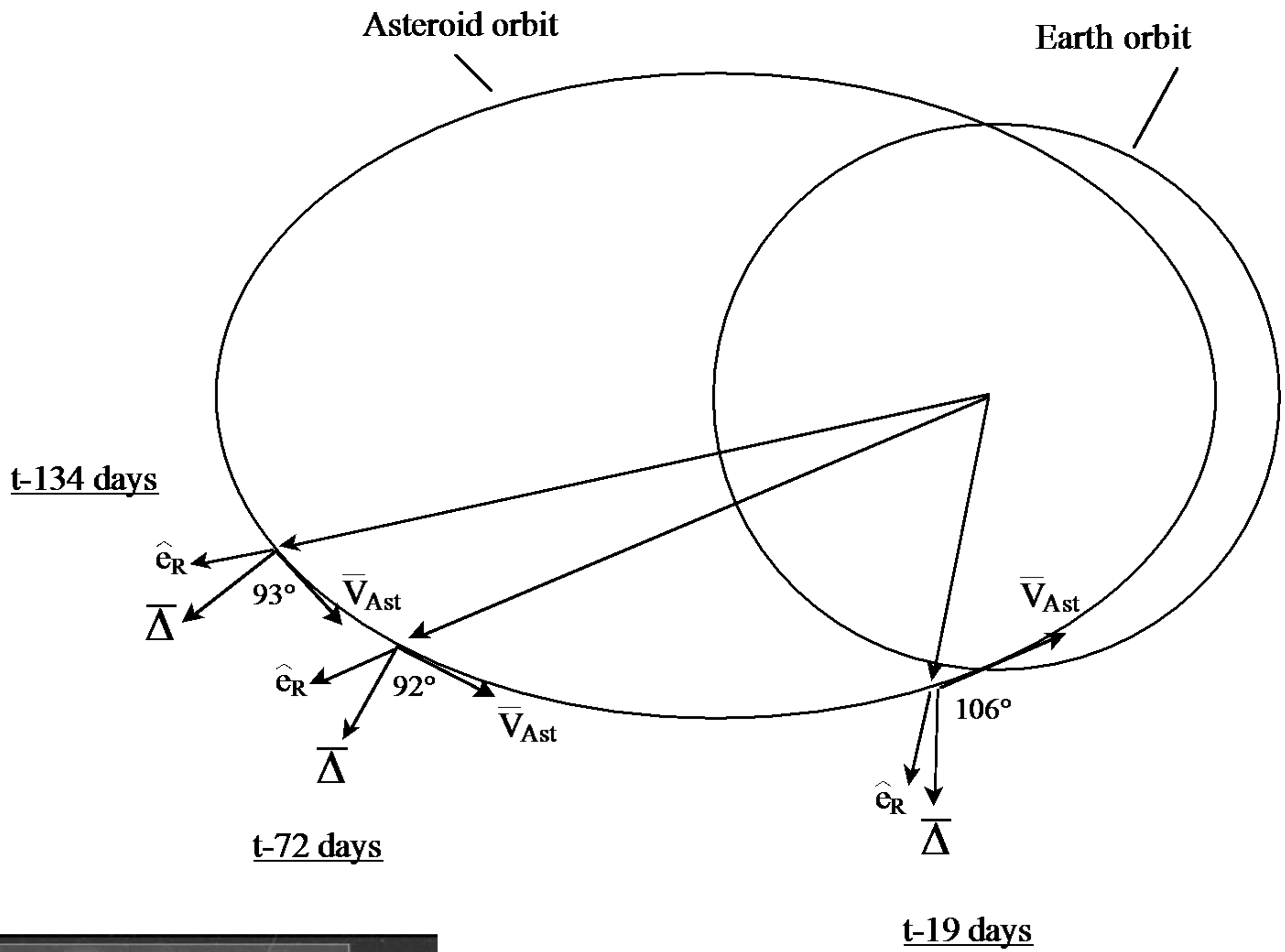


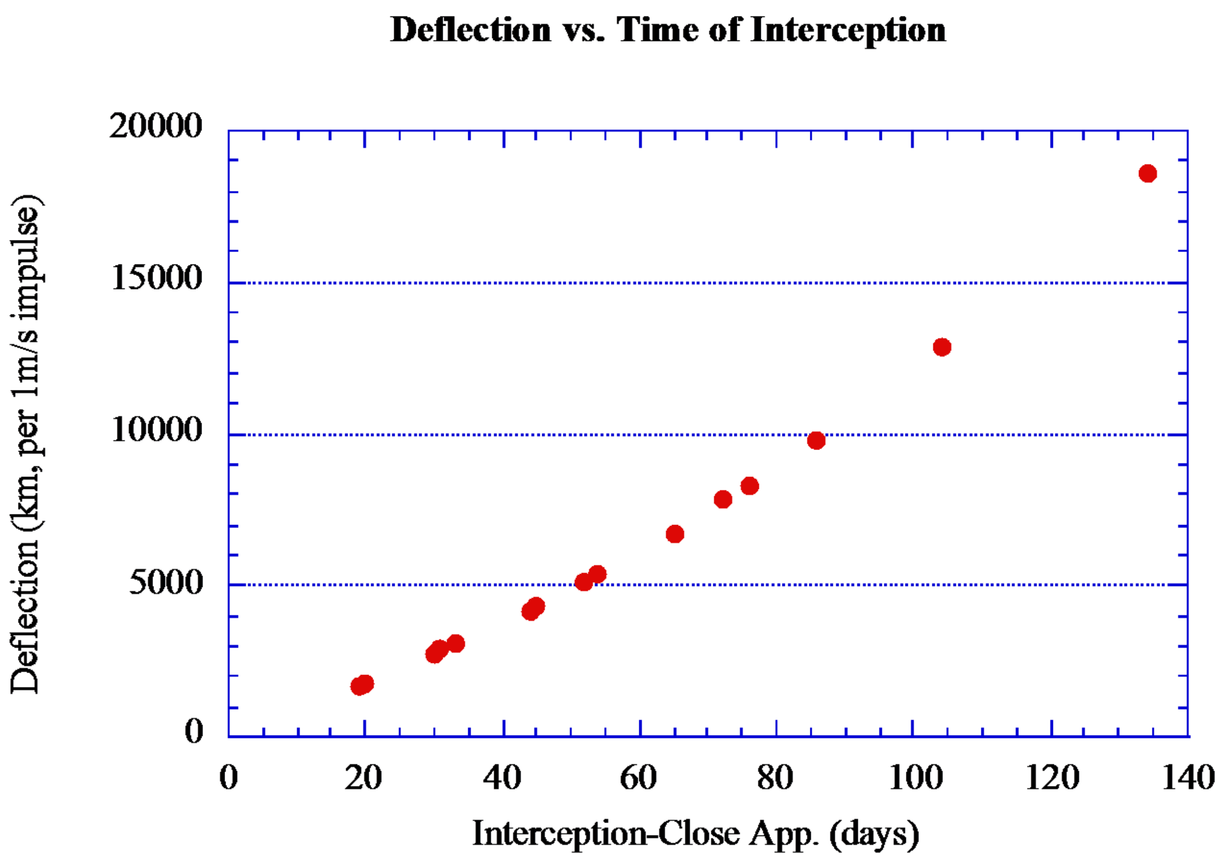
In-plane (azimuthal) angle



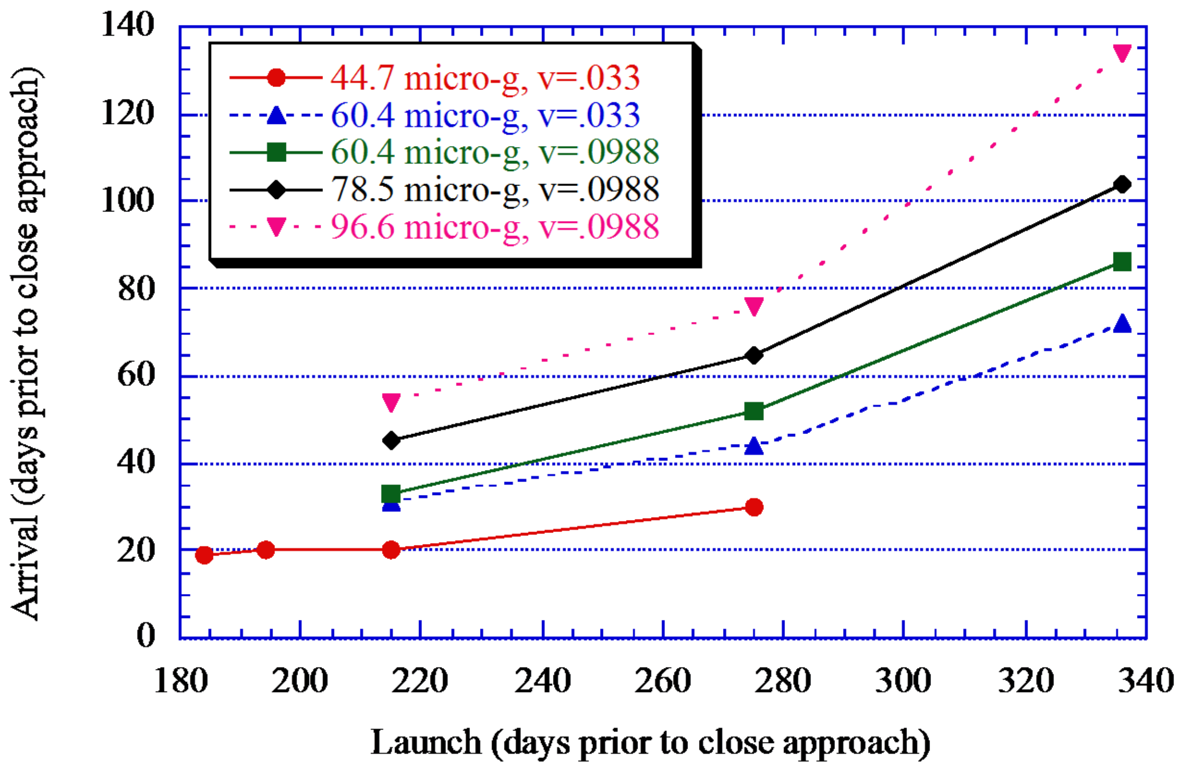
Out-of-plane angle

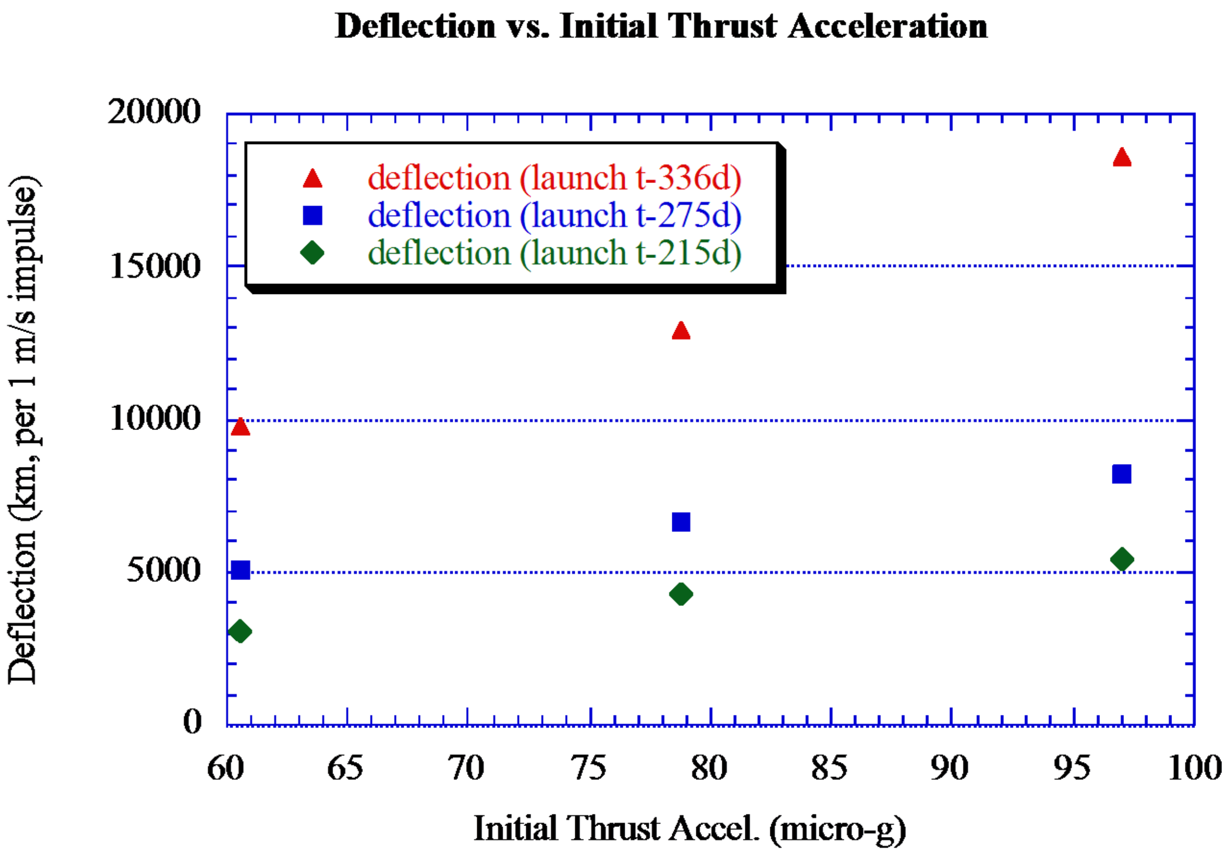
Direction of Optimal Nuclear Deflection Impulse

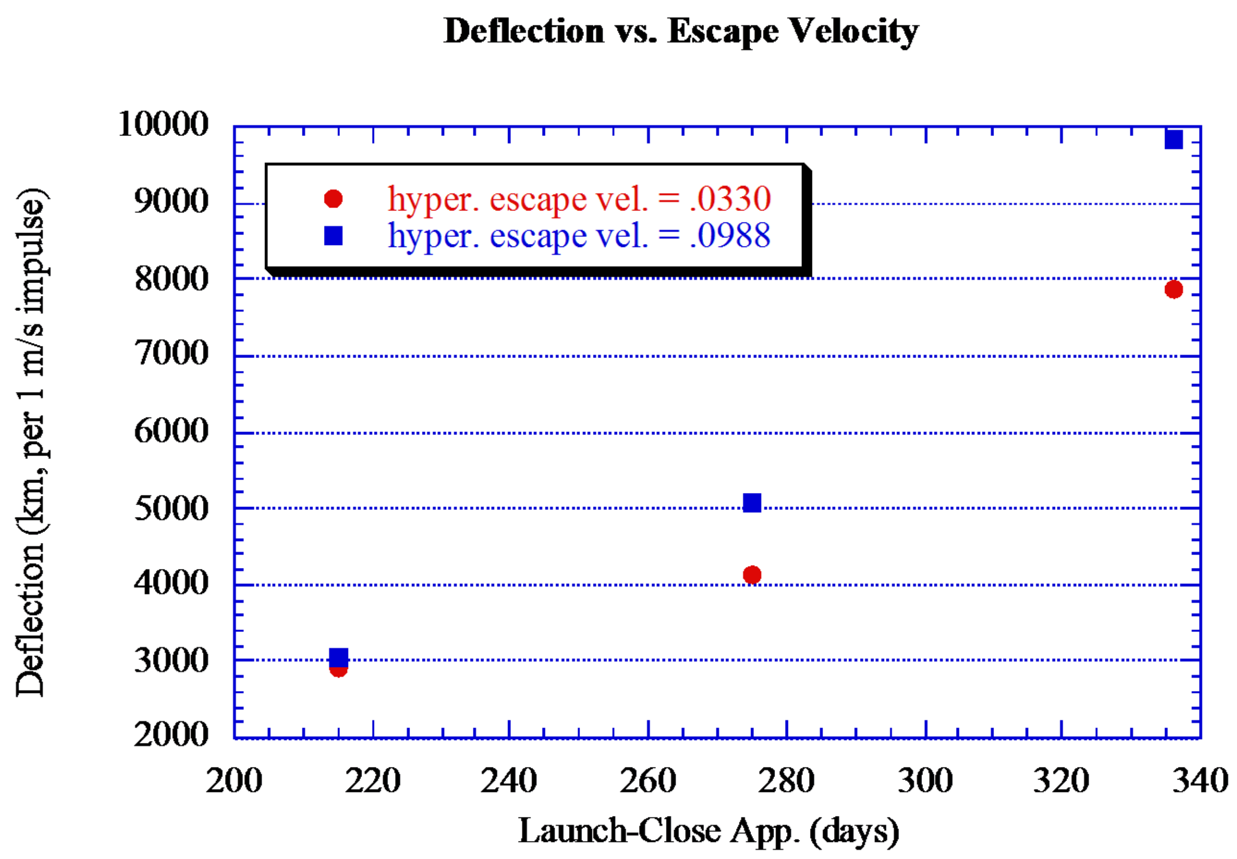



Deflection vs. Time of Interception
Deflection (km, per 1m/s impulse)
20000
15000
10000
5000
0
0
20
40
60
80
100
120
140
Interception-Close App. (days)

Arrival Time vs. Launch Time




Deflection vs. Initial Thrust Acceleration
20000
15000
10000
5000
0
60
65
70
75
80
85
90
95
100
Initial Thrust Accel. (micro-g)
Deflection (km, per 1 m/s impulse)
deflection (launch t-336d)
deflection (launch t-275d)
deflection (launch t-215d)


Deflection vs. Escape Velocity
hyper. escape vel. = .0330
hyper. escape vel. = .0988
10000
9000
8000
7000
6000
5000
4000
3000
2000
200
220
240
260
280
300
320
340
Deflection (km, per 1 m/s impulse)
Launch-Close App. (days)

Advantage of Interception Using Low-Thrust (1)

Compare propellant and structural mass fractions required to intercept asteroid, in same time-of-flight, for impulsive case vs. using low-thrust electric propulsion.

Uses results from a typical case.

Problem: Find DV required to take spacecraft from given $\bar{r}_1$ to $\bar{r}_2$ in flight time t_F = 2.5096 units = 145.89 days.

Method: This is a "Lambert" problem. Solve Lambert's time-of-flight equation for the semimajor axis of the elliptical section connecting the given points in the specified time.

Result: Trajectory has a = 1.31973 AU ; can solve for absolute velocity at departure yielding $\bar{v}_1 = [\ .0931, -1.040, -.3883\ ]$

Velocity at escape from Earth is $\bar{v}_{Earth} + \bar{v}_{¥} = [\ -.0241, -1.008, -0.020\ ]$

Difference is required $\overline{DV} = [\ .069, -.032, -.368\ ]$

Advantage of Interception Using Low-Thrust (2)

Then $\left|\overline{DV}\right|$ = .414 in normalized units
= 12.32 km/sec

For a optimal two-stage chemical rocket, with (good) Isp = 375 sec,
and structural coefficient of 0.12 for each stage:
1st stage mass = 70.53 x payload mass
2nd stage mass = 7.91 x payload mass
thus, propellant required is 88% of rocket mass or 69.0 x payload mass.

Payload mass is approximately 1-3% of mass at departure!

For the low-thrust (electric) motor, with Isp = 4000 sec, of case N, final thrust acceleration is .191, initial thrust accel. is .14 = .83 mN/kg thus,

$$\frac{m_0}{m_0 - m_{fuel}} = \frac{\text{final acceleration}}{\text{initial acceleration}} = \frac{.191}{.140} = 1.364$$

i.e., fuel mass is 27% of payload mass (assuming structural coefficient $e = 0.1$)
Present ion propulsion technology requires 1.4 kg propulsion hardware/mN thrust. Near-term improvement expected to 0.7 kg/mN. Thus total propulsion system mass is .83mN(.7kg/mN) + .27kg fuel + .03 kg tank = .88kg/kg. So payload mass is approximately 12% of mass at departure!

Example: Deflection of PHA Using Impact and a Low-Thrust Spacecraft

J. Englander M.S. thesis, Univ. of Illinois (2008)

Objective is to find a low-thrust trajectory for a spacecraft whose impact is to maximize the subsequent deflection <u>distance</u> when the asteroid approaches the Earth.

Launch vehicle upper stage applies a departure impulse in low Earth orbit. The amount of fuel used for this purpose (at lower I_{sp}) is chosen by the optimizer. Then the spacecraft engages a low thrust electric motor and travels to the asteroid.

The time histories of thrust magnitude and steering angle (the two continuous controls) must be found to optimally guide the spacecraft to the asteroid.

An approximate optimal (and feasible) solution is found with a GA using a "shape-based" representation of the trajectory. The GA has 9 free parameters: it uses 50 generations with a population n = 50.

The solution from the GA is then used to initialize a much more accurate solution using a direct transcription method (Runge-Kutta parallel shooting) and a NLP problem solver (SNOPT)

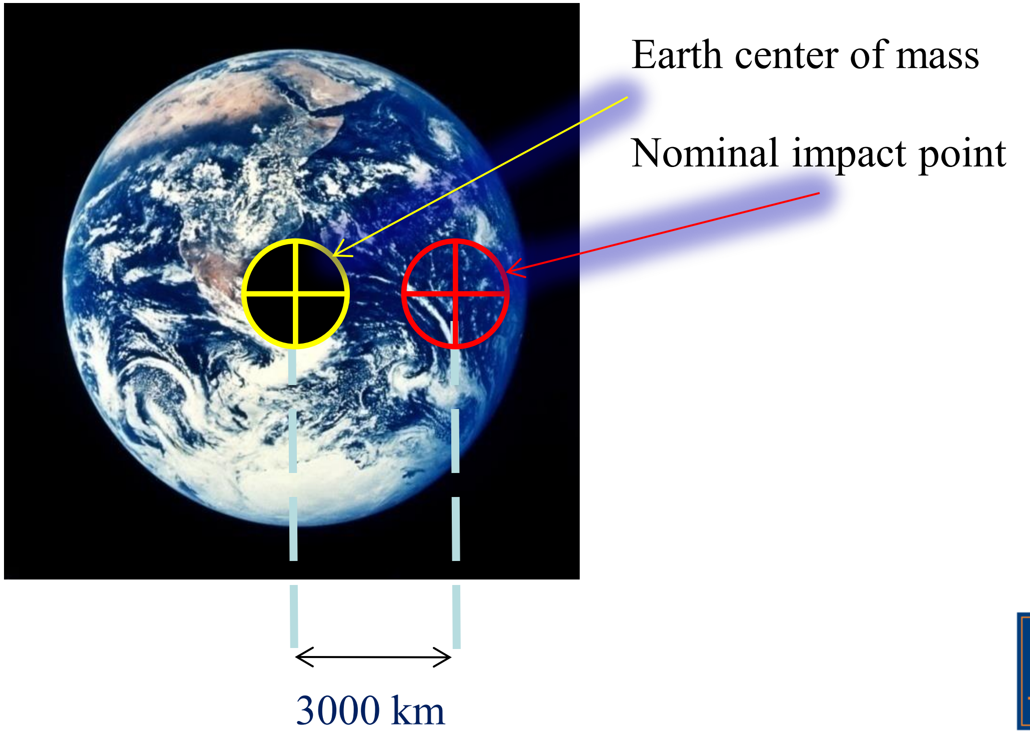
The Asteroid

- Hypothetical asteroid based on 99942 Apophis impacts Earth on April 13th, 2029 (Apophis will miss Earth by 32000 km)

- Same mass as 99942 Apophis (~4.6×10^{10} kg)

Orbit elements , April 13th, 2021 (very slightly modified to yield impact rather than a 32000 km miss):

$a = 0.9214$ AU
$e = 0.1957$
$i = 3.42^{o}$
$\omega = 126.62^{o}$
$\Omega = 203.79^{o}$
$f = 231.54^{o}$



Assumptions

- Spacecraft is launched on Delta IV Heavy
- Delta IV heavy can place 25000 kg in 300km altitude LEO parking orbit, including spacecraft, fuel, and launch vehicle upper stage dry mass
- Upper stage engine has Isp = 462s (RL-10)
- Low thrust electric motor (same as Dawn, DS-1) has Isp = 3100s, thrust = 90mN
  - We mount two of these engines, for a total thrust of 180mN
- Launch window opens April 13th, 2021 – 8 years before the 2029 impact

The Optimal Trajectory Parameters Found via GA

GA parameters and their bounds

Parameter	Lower bound	Upper bound
Launch date after epoch (TU)	0.001	8.40
Flight time (TU)	0.001	40.0
Number of revolutions	0.0	1.0
Arrival velocity magnitude (AU/TU)	0.001	5.0
In-plane arrival flight path angle (radians)	$-\pi/4$	$\pi/4$
Out-of-plane arrival flight path angle (radians)	$-\pi/4$	$\pi/4$
Initial impulse in-plane pointing angle (radians)	$-\pi$	π
Initial impulse out-of-plane pointing angle (radians)	$-\pi$	π
Propellant used for initial impulse (kg)	12985	19000

Optimal values of the GA parameters

Parameter	Optimal value
Launch date after epoch (TU)	4.8705
Flight time (TU)	10.2041
Number of revolutions	0.1267
Arrival velocity magnitude (AU/TU)	1.2180
In-plane arrival flight path angle (radians)	0.1916
Out-of-plane arrival flight path angle (radians)	0.0130
Initial impulse in-plane pointing angle (radians)	0.7938
Initial impulse out-of-plane pointing angle (radians)	-3.0736
Propellant used for initial impulse (kg)	14773

Shape-Based Approximation to Trajectory

The 9 GA parameters determine (among other things) $\left[r_1,\ \theta_1,\ \dot{r}_1,\ \dot{\theta}_1\right]$ and $\left[r_2,\ \theta_2,\ \dot{r}_2,\ \dot{\theta}_2\right]$

Then the path is approximated parametrically as an inverse 6th degree polynomial:

$$r(\theta) = \frac{1}{a + b\theta + c\theta^2 + d\theta^3 + e\theta^4 + f\theta^5 + g\theta^6}$$

The seven coefficients *a* – *g* may be solved for using the 9 GA parameters directly and indirectly, e.g. it is obvious that $r_1 = 1/a$.

The thrust magnitude and flight path angle may be found *a posteriori* as

$$\Gamma_{in} = -\frac{\mu}{2r^3\cos\alpha}\,\frac{6d + 24e\theta + 60f\theta^2 + 120g\theta^3 - (\tan\alpha)/r}{\left[(1/r) + 2c + 6d\theta + 12e\theta^2 + 20f\theta^3 + 30g\theta^4\right]^2}$$

$$\tan\alpha = -r\cdot\left(b + 2c\theta + 3d\theta^2 + 4e\theta^3 + 5f\theta^4 + 6g\theta^5\right)$$

For the 3D case need to also parameterize the vertical motion using:

$$z(\theta) = a_z + b_z\theta + c_z\theta^4 + d_z\theta^5$$

GA Convergence History

GA objective function is scaled by parameter h in order to penalize trajectories that need a thrust acceleration larger than what the vehicle can actually provide, i.e.

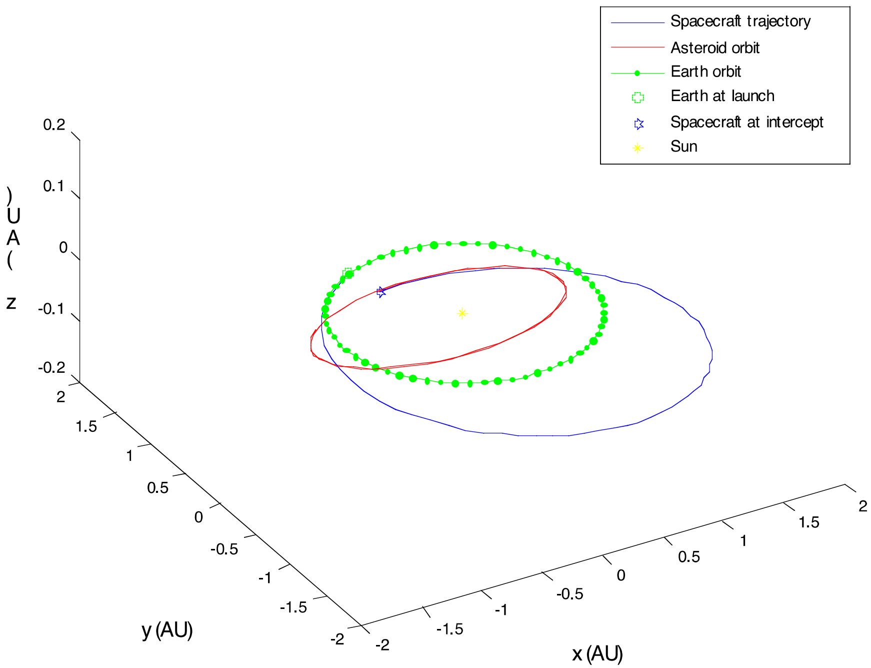
$$J = - r_{miss} \times h$$

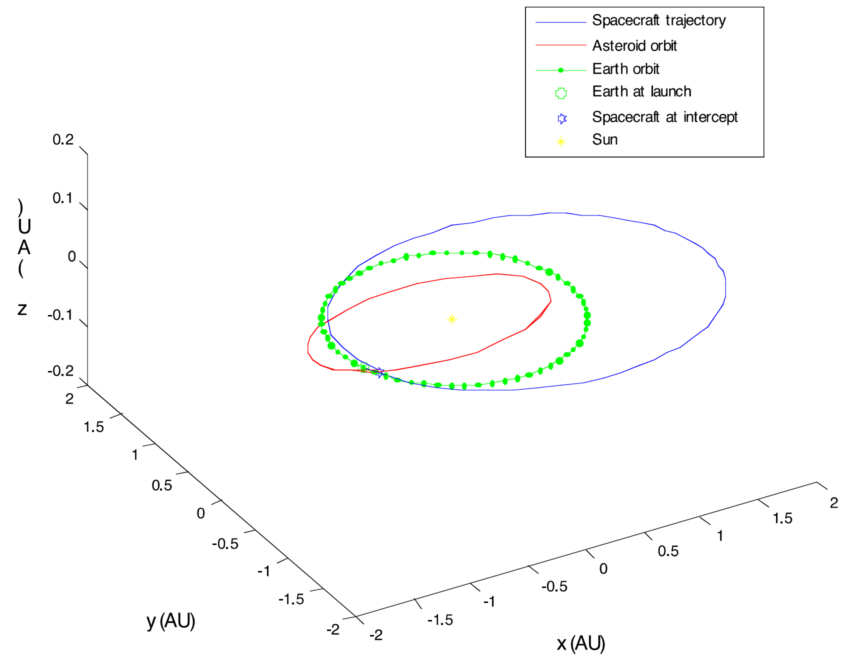
$$h = \frac{\left(NT_{\max} \big/ m_{initial} \right)}{G_{average}}$$

Generation	Best J (km)
1	-58.1
2	-94.9
3	-94.9
4	-333.5
5	-609.4
7	-859.6
10	-1257
12	-1318
14	-1483
18	-1729
21	-2408
26	-2519
27	-2959
28	-3387
29	-4048
37	-4496
38	-4754
39	-4943
40	-5162
41	-5178
47	-5358
50	-5358

Þ equiv. to 30,967 km

Guess Trajectory (from GA) and Final Trajectory





Initial guess of 3D trajectory from the GA

Converged NLP solution using this initial guess

Converged Solution From NLP Solver

- Departure: 13 April 2021

  $\Delta V = 5.36\ km/\sec$

  Consumes 15613 kg of propellant

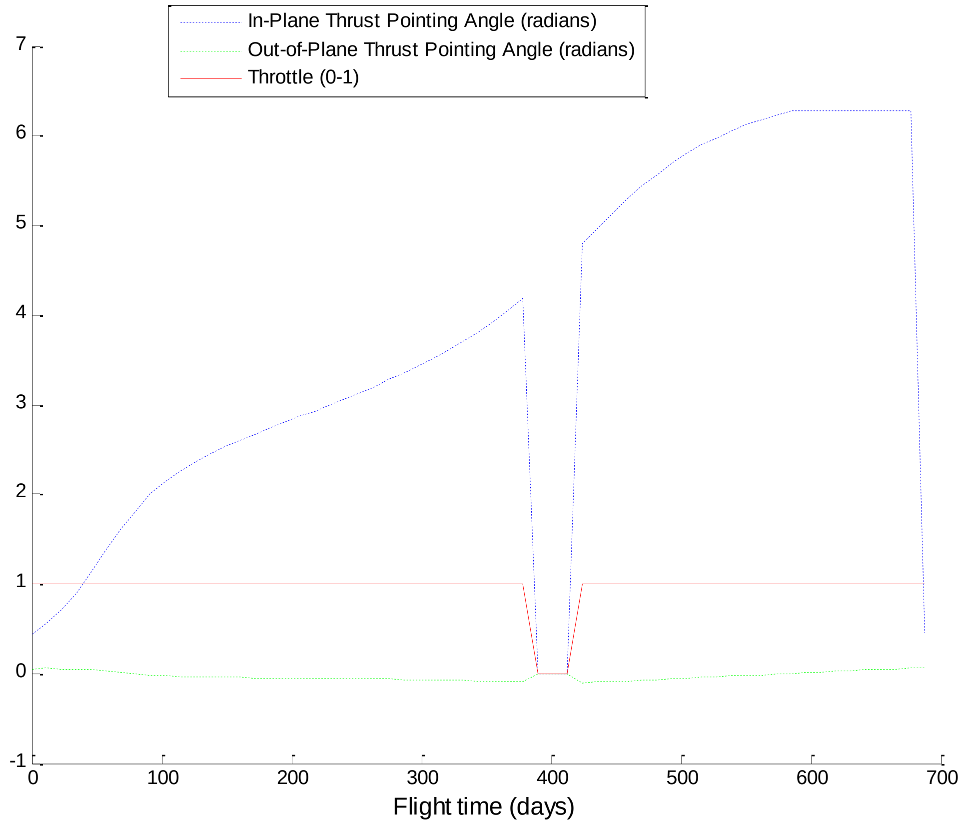
- Powered flight: Consumes 696 kg of propellant

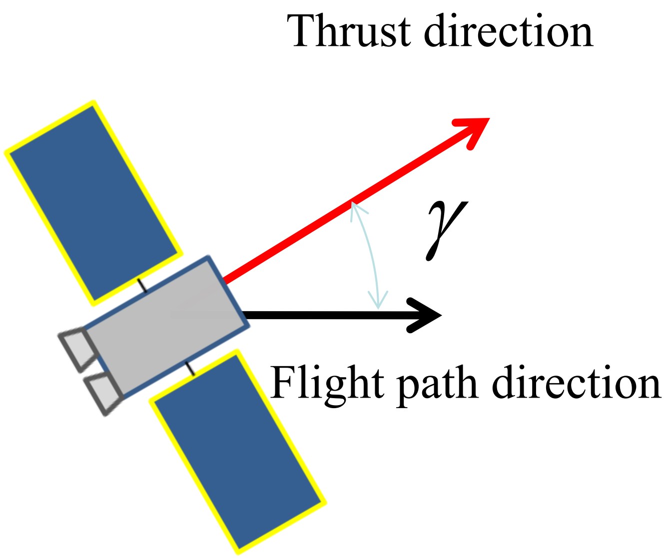
- Interception: 2 March 2023

  Impact changes velocity of Apophis by 2.7 mm/sec

- Deflection: 17041 km

Optimal Control Time History





Example: Manned Asteroid Sample Return Mission with Time Constraint (1)

Aishwarya Stanley, M.S. thesis 2013

Objective is to minimize ΔV for sample return mission with 8-25 day stay at asteroid and max 365 day total trip time.

NEA catalog searched for candidates satisfying the following criteria:

1. Allow departure dates between 2025 and 2035
2. Are at least 30 m in diameter
3. Allow 365 day round trip missions

and are not limited by the following considerations:

4. Uncertain orbit and/or limited Earth-based observation
5. Few departure opportunities
6. Likely too small based on estimated albedo (albedo assumed to be between 0.05 and 0.25)

Example: Manned Asteroid Sample Return Mission with Time Constraint (2)

Lambert's method is used to find the ΔV's required for Earth departure, asteroid interception, asteroid departure, and Earth arrival.

The only free parameters are the four dates of those events.

NASA researchers had used a brute force approach, generating tens of thousands (hundreds of thousands?) of missions for each asteroid for various values of each of those 4 dates.

Our research used PSO to determine the optimal mission.

A penalty function is used to limit total flight time to 365 days.

Example: Manned Asteroid Sample Return Mission with Time Constraint (3)

Asteroid Name	Pop, Gen	Lower Bounds	Upper Bounds	Desired Epoch Date, t	Total Delta-V (km/s)	Optimal t_launch, days	Optimal Launch Date	Optimal t_flight1, t_flight2 days	Optimal t_wait, days	Total Mission Duration, days
99942 Apophis	100, 2000	[30,90,8,100]	[3000,782,25,782]	Jan 1, 2023	7.4688	2757.2	July 20, 2030	198.7 138.9	25	362.6
99942 Apophis	100, 2000	[30,90,8,100]	[3000,782,25,782]	Sep 30, 2022	7.6139	2850.2	July 20, 2030	202.9 146.3	12.7	361.9
2011 AA 37	100, 2000	[60,90,8,90]	[1000,782,25,782]	Jan 18, 2028	7.2843	88.2058	Apr 15, 2028	173.6769 175.6233	14.6220	363.9222
2011 AA 37	100, 2000	[60,90,8,90]	[1000,782,25,782]	Apr 18, 2027	7.1332	361.2868	Apr 13, 2028	184.2333 172.6753	8.0165	364.9251
2007 YF	100, 2000	[30,80,8,80]	[4000,782,25,782]	Apr 18, 2021	6.1038	3464.8	Oct 13, 2030	154 83.7	17.6	255.3
2007 YF	100, 2000	[30,80,8,80]	[800,782,25,782]	Sep 1, 2029	5.9626	407.6332	Oct 14, 2030	156.2895 90.8209	8.0005	255.1109
2009 CV	100, 2000	[30,90,8,100]	[4000,782,25,782]	Apr 18, 2023	9.1055	606.6053	Dec 15, 2024	190.7190 134.5491	24.9987	350.2668
2009 CV	100, 2000	[30,90,8,100]	[300,782,25,782]	June 1, 2024	9.1052	196.9495	Dec 15, 2024	190.7593 134.7199	24.9998	350.479

Example: Manned Asteroid Sample Return Mission with Time Constraint

Name	Gen	Bounds		Epoch Date, t	Delta-V (km/s)	t_launch, days	Launch Date	t_flight1, t_flight2 days	t_wait, days	Mission Duration, days
2007 UY 1	100, 2000	[105,90,8,100]	[800,782,25,782]	Apr 18, 2030	6.0337	422.8191	June 15, 2031	135.9333 221.1177	8.0006	365.0513
2007 UY 1	100, 2000	[60,90,8,100]	[1000,782,25,782]	Dec 30, 2029	6.0306	532.6974	June 16, 2031	137.5176 219.5918	8.0002	365.1096
1999 AO 10	100, 2000	[60,90,8,100]	[1000,782,25,782]	Apr 18, 2023	8.2821	194.5723	Oct 30, 2023	126.7320 148.6869	8.7617	284.1806
1999 AO 10	100, 2000	[60,90,8,100]	[1000,782,25,782]	Aug 1, 2022	8.2821	454.7638	Oct 30, 2023	126.7320 148.6869	8.7617	284.1806
2001 CQ 36	100, 2000	[60,90,8,100]	[1000,782,25,782]	Apr 18, 2028	7.2320	229.4854	Dec 3, 2028	124.7767 151.9676	15.8906	292.6349
2001 CQ 36	100, 2000	[30,90,8,100]	[1000,782,25,782]	Oct 18, 2028	7.2108	46.4626	Dec 3, 2028	124.5903 144.6229	24.9992	294.2124
1999 CG 9	100, 2000	[60,90,8,100]	[3000,782,25,782]	Apr 18, 2023	5.4828	954.3529	Nov 27, 2025	156.3530 188.5806	18.9328	363.8664
1999 CG 9	100, 2000	[30,90,8,100]	[3000,782,25,782]	June 1, 2023	5.4796	910.5816	Nov 28, 2025	156.6384 188.5814	18.2344	363.4542